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Original Research Article

Closed-Form Optimality Systems and Physically Weighted Linear–Quadratic Control for φ-Conformable Dynamics

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Abstract: We study finite-horizon optimal control problems governed by the local φ-conformable derivative introduced by Sadek. The central observation is an exact operational-time transformation, under which a φ-conformable differential equation becomes an ordinary differential equation. This observation is used to prove inverse, integration-by-parts and calculus identities, give a detailed direct-method existence proof, derive necessary and sufficient Hamiltonian optimality conditions, and matrix-exponential closed-form solutions of autonomous linear–quadratic problems with prescribed or free terminal states. We distinguish two cost measures that are often conflated: the natural φ-conformable measure and physical time. For physicaltime costs we obtain the correct weighted canonical equations, a nonautonomous Riccati differential equation, existence and uniqueness results, and comparison bounds for optimal values. A weighted controllability Gramian yields a closed-form physical-time minimum-energy law and an explicit sharp nonattainment threshold for singular clocks. Exact scalar solutions and independently checked numerical examples illustrate the effect of fractional order and the choice of the reference function.

Keywords: φ-conformable derivative; optimal control; Riccati equation; transversality.

1. Introduction

The conformable approach to fractional-order models replaces the usual derivative by a local, order-dependent differential operator. Chiranjeevi and Biswas [1] studied closed-form solutions of two scalar conformable optimal control problems with fixed terminal time. The generalized φ-conformable calculus was introduced by Sadek [2]. Standard conformable linear–quadratic control and Riccati techniques had already been developed, including in Sadek and collaborators [3]; conformable differential-transform methods for optimal control were studied subsequently [4]. For conformable control problems and their generalizations, there are also numerical gradient methods [5, 6]. These precedents should not be confused with the following results. Nonlocal ψ-Caputo fractional optimal control problems have also been considered [7], but those operators have quite distinct analytic properties. The central mathematical point is the importance of the chosen measure. Complete and selfcontained proofs are given for all principal identities of calculus and control used. The use of the natural measure for the corresponding local φ-derivative leads to the standard optimal control problem on a changed time scale, while the use of the physical time results in the appearance of a

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nonconstant weight factor and the corresponding optimal control and cost. This approach is helpful for the formulation of models but cannot alone provide a new phenomenon of nonlocal nature. All claims made in the present paper are proven under specified assumptions about the spaces of functions and the terminal points; we replace every appeal to textbooks where it is possible by the direct proof. We thus (i) specify the equivalent classical problem, (ii) prove the full optimality conditions, (iii) consider fixed and free terminal points, and (iv) find the modified physical-time Riccati system.

Classical conformable calculus, conformable variational principles, and linear–quadratic control are discussed in [8–11]. Complementary studies address different fractional settings: Derbazi et al. [12] obtained extremal solutions for generalized Caputo-type boundary-value problems by monotone iteration; Redhwan et al. [13] studied implicit Caputo–Katugampola equations with anti-periodic conditions; Almalahi et al. [14] modeled 2019-nCoV transmission with a fractal–fractional operator. Unlike these nonlocal formulations, the local φ-conformable derivative in this paper admits an exact operational-time transformation. This transformation also makes explicit the mathematical distinction between natural-clock and physical-time cost measures; no nonlocal existence or stability results are invoked in deriving our optimality systems.

2. The precise φ-conformable calculus

Let $a < T$, $0 < \alpha \leq 1$, and suppose $\varphi : [a,T] \to R$ is continuous, strictly increasing, C^1 on $(a,T]$, and $\varphi'(t) > 0$, for $t > a$. Put

$$g(t) = \varphi(t) - \varphi(a), \qquad h(t) = \frac{g(t)^\alpha}{\alpha}, \qquad q(t) = h'(t) = \varphi'(t) g(t)^{\alpha-1} > 0 \quad (t > a). \tag{1}$$

The symbol q is reserved for the clock density; state penalties will be denoted by Q. Integrals over (a,T) are understood as Lebesgue integrals. In particular, q need not remain bounded as $t \downarrow a$.

Lemma 2.1 (Clock regularity and integrability). Under the stated hypotheses, $h \in \mathrm{AC}([a,T])$, its almost-everywhere derivative is q, and

$$\int_a^T q(t)\,\mathrm{d}t = S = \frac{[\varphi(T) - \varphi(a)]^\alpha}{\alpha} < \infty. \tag{2}$$

Furthermore h is strictly increasing, has a continuous inverse from $[0,S]$ to $[a,T]$, and is locally a C^1 diffeomorphism on $(a,T]$.

Proof. The chain rule for ordinary differentiable functions gives $h'(t) = \alpha^{-1}\alpha g(t)^{\alpha-1}\varphi'(t) = q(t)$ whenever $t > a$. For every $b \in (a,T)$ the fundamental theorem of calculus applied on $[b,T]$ yields $h(T) - h(b) = \int_b^T q(t)\,\mathrm{d}t$. As $b \downarrow a$, continuity gives $h(b) \to h(a) = 0$ and monotone convergence, because $q > 0$, gives (2). Similarly $h(t) = \int_a^t q(s)\,\mathrm{d}s$, proving absolute continuity directly from the L^1 representation. Strict increase follows from $g(t) > 0$, $\alpha > 0$, and strict increase of φ. A continuous, strictly monotone function on a compact interval has a continuous inverse. On each compact interval inside $(a,T]$, $h'(t) = q(t) > 0$ and the usual inverse function theorem applies. The claim does not require a finite positive value of $q(a)$. □

Definition 2.2 (φ-conformable derivative [2]). For $t \in (a,T]$, define

$${}_a\mathcal{C}^{\alpha,\varphi} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon/q(t)) - f(t)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{f\left(t + \dfrac{\alpha\varepsilon}{\frac{\mathrm{d}}{\mathrm{d}t}[\varphi(t) - \varphi(a)]^\alpha}\right) - f(t)}{\varepsilon}, \tag{3}$$

whenever the limit exists. The integral associated with the same clock is

$$ {}_a\mathcal{I}^{\alpha,\varphi} f(t) = \int_a^t f(s) q(s)\, \mathrm{d}s. \tag{4} $$

Proposition 2.3 (Differential and inverse formulas). If f is differentiable at $t > a$, then

$$ {}_a\mathcal{C}^{\alpha,\varphi} f(t) = \frac{f'(t)}{q(t)} = \frac{[\varphi(t) - \varphi(a)]^{1-\alpha}}{\varphi'(t)} f'(t). \tag{5} $$

If f is absolutely continuous, then ${}_a\mathcal{J}^{\alpha,\varphi}({}_a\mathcal{C}^{\alpha,\varphi} f)(t) = f(t) - f(a)$. If f is continuous for $t > a$ and integrable with respect to $q(t)\mathrm{d}t$, then ${}_a\mathcal{C}^{\alpha,\varphi}({}_a\mathcal{J}^{\alpha,\varphi} f)(t) = f(t)$ for $t > a$.

Proof. Fix $t \in (a, T)$, so $q(t)$ is finite and strictly positive. Put $\delta_\varepsilon = \varepsilon / q(t)$. Then $\delta_\varepsilon \to 0$ and

$$ \frac{f(t+\delta_\varepsilon) - f(t)}{\varepsilon} = \frac{f(t+\delta_\varepsilon) - f(t)}{\delta_\varepsilon} \frac{\delta_\varepsilon}{\varepsilon} \longrightarrow f'(t) \frac{1}{q(t)}. $$

The formula for $1/q(t)$ follows by substituting (1). At $t = T$ the difference quotient must be interpreted using increments that stay in the interval. Next let $f \in \mathrm{AC}([a, T])$. Its a.e. ordinary derivative belongs to L^1, and at each interior point where f' exists we have ${}_a\mathcal{C}^{\alpha,\varphi} f = f'/q$. Therefore the product $({}_a\mathcal{C}^{\alpha,\varphi} f) q = f'$ a.e., and

$$ {}_a\mathcal{I}^{\alpha,\varphi}({}_a\mathcal{C}^{\alpha,\varphi} f)(t) = \int_a^t ({}_a\mathcal{C}^{\alpha,\varphi} f)(s) q(s)\, \mathrm{d}s = \int_a^t f'(s)\, \mathrm{d}s = f(t) - f(a). $$

The condition that the φ-derivative need not be defined at a is immaterial for the Lebesgue integral. Conversely, assume $qf \in L^1(a, T)$ and f is continuous at a fixed interior point $t > a$. Set $F(t) = {}_a\mathcal{J}^{\alpha,\varphi} f(t)$. The function qf is continuous at that point because q is continuous on $(a, T]$. The fundamental theorem for an integral with an integrand continuous at t gives $F'(t) = q(t) f(t)$. By the first part, ${}_a\mathcal{C}^{\alpha,\varphi} F(t) = F'(t)/q(t) = f(t)$. If f is only measurable, the same conclusion holds almost everywhere by the Lebesgue differentiation theorem, but not necessarily pointwise.

Theorem 2.4 (Exact time-change equivalence). Let $S = h(T) > 0$. The map $h\colon [a, T] \to [0, S]$ is a continuous increasing bijection. For $t > a$ and $\tau = h(t)$ set $y(\tau) = x(h^{-1}(\tau))$. If x is absolutely continuous in the h-clock and its transformed representative y is absolutely continuous, then

$$ {}_a\mathcal{C}^{\alpha,\varphi} x(t) = y'(h(t)) \quad a.e., \qquad \int_a^T F(t) q(t) \mathrm{d}t = \int_0^S F(h^{-1}(\tau)) \mathrm{d}\tau. \tag{6} $$

In particular ${}_a\mathcal{C}^{\alpha,\varphi} x(t) = f(t, x(t), u(t))$ is equivalent to $y'(\tau) = f(h^{-1}(\tau), y(\tau), v(\tau))$, with $v(\tau) = u(h^{-1}(\tau))$.

Proof. Lemma 2.1 shows that h maps $[a, T]$ bijectively onto $[0, S]$. Write $t(\tau) = h^{-1}(\tau)$ and take $y \in \mathrm{AC}([0, S]; \mathbb{R}^n)$ with $x(t) = y(h(t))$. Because h is absolutely continuous and monotone, composition and the chain rule give, almost everywhere on (a, T),

$$ x'(t) = y'(h(t)) h'(t) = q(t) y'(h(t)). $$

Consequently

$$ {}_a\mathcal{C}^{\alpha,\varphi} x(t) = x'(t)/q(t) = y'(h(t)), $$

almost everywhere on the domain of these derivatives. It is clear what the converse means for the case of the clock-absolutely-continuous class: that the representative $x \circ h^{-1}$ belongs to $AC([0, S])$. Such an agreement prevents the wrong statement about absolute continuity of arbitrary trajectories in physical time under any singular inverse clock.

Now we can apply the monotone change of variables theorem to $F \geq 0$ on $[a + \delta, T]$, where the usual formula holds. Now let $\delta \to 0$ using monotone convergence. The proof is extended to positive and negative parts for a signed function. Hence, $\int_a^T F(t)q(t)dt = \int_0^S F(t(\tau))d\tau$, whenever either of the absolute integrals converges.

Finally, substituting $x = y \circ h$, $u = v \circ h$ and the derivative identity into ${}_a\mathcal{C}^{\alpha,\varphi}x = f(t, x, u)$ gives $y'(\tau) = f(t(\tau), y(\tau), v(\tau))$ almost everywhere. Reversing the substitutions proves the converse, with identical initial and terminal states because $h(a) = 0$ and $h(T) = S$. □

Proposition 2.5 (φ-integration by parts). For functions v, w absolutely continuous in the operational time τ,

$$\int_a^T v(t)^{\mathsf{T}}\, {}_a\mathcal{C}^{\alpha,\varphi}w(t)\, q(t)\mathrm{d}t = \left[v(t)^{\mathsf{T}}w(t)\right]_a^T - \int_a^T ({}_a\mathcal{C}^{\alpha,\varphi}v(t))^{\mathsf{T}}w(t)q(t)\mathrm{d}t. \tag{7}$$

Proof. Define $V(\tau) = v(h^{-1}(\tau))$ and $W(\tau) = w(h^{-1}(\tau))$. Both are absolutely continuous by hypothesis. The time-change theorem gives

$$\int_a^T v(t)^{\mathsf{T}}\, {}_a\mathcal{C}^{\alpha,\varphi}w(t)q(t)\,\mathrm{d}t = \int_0^S V(\tau)^{\mathsf{T}}W'(\tau)\,\mathrm{d}\tau.$$

The ordinary product rule, applied to the absolutely continuous scalar function $V^{\mathsf{T}}W$, says $(V^{\mathsf{T}}W)' = (V')^{\mathsf{T}}W + V^{\mathsf{T}}W'$ almost everywhere. Integrate from 0 to S and rearrange:

$$\begin{aligned}\int_0^S V^{\mathsf{T}}W'\,\mathrm{d}\tau &= [V^{\mathsf{T}}W]_0^S - \int_0^S (V')^{\mathsf{T}}W\,\mathrm{d}\tau \\ &= [v^{\mathsf{T}}w]_a^T - \int_a^T ({}_a\mathcal{C}^{\alpha,\varphi}v)^{\mathsf{T}}wq\,\mathrm{d}t.\end{aligned}$$

All displayed terms are integrable because V, W are bounded and their derivatives belong to L^1. Thus no unproved boundary cancellation at a possibly singular $q(a)$ is used.

Proposition 2.6 (The correct control spaces under time change). Let $u\colon [a, T] \to \mathbb{R}^m$ be measurable and put $v(\tau) = u(h^{-1}(\tau))$. Then

$$\begin{aligned}\int_a^T \|u(t)\|^2 q(t)\,\mathrm{d}t &= \int_0^S \|v(\tau)\|^2\mathrm{d}\tau, \\ \int_a^T \|u(t)\|^2\mathrm{d}t &= \int_0^S \|v(\tau)\|^2 w(\tau)\mathrm{d}\tau.\end{aligned} \tag{8}$$

Thus $L^2([a, T], q\mathrm{d}t)$ is isometric to $L^2(0, S)$, while the physical-time control class is isometric to the weighted space $L^2((0, S), w\mathrm{d}\tau)$. The two classes agree with equivalent norms if $0 < c \leq w \leq C < \infty$.

Proof. Apply the positive-function version of the measure substitution in Theorem 2.4 to $F(t) = \| u(t) \|^2$. This proves the first formula, including the possibility that both sides are infinite. In the second integral write $\mathrm{d}t = w(h(t))q(t)\mathrm{d}t$ since $w(h(t)) = 1/q(t)$, and apply the same substitution to $\| u(t) \|^2\, w(h(t))$. If $c \leq w \leq C$, the elementary inequalities $c \| v \|_{L^2}^2 \leq \| v \|_{L^2(w\mathrm{d}\tau)}^2 \leq C \| v \|_{L^2}^2$ prove both equality of the underlying sets and norm equivalence. Without such bounds the two classes may differ: for $a = 0$, $\alpha = 1/2$, and $\varphi(t) = t$, one has $q(t) = t^{-1/2}$, so $u(t) = t^{-3/10}$ lies in physical-time $L^2(0,1)$ but not in $L^2((0,1), q\mathrm{d}t)$. Indeed its physical integrand is $t^{-3/5}$ (integrable near zero) while its clock-

weighted integrand is $t^{-11/10}$ (not integrable). This example rules out treating the two control spaces as automatically interchangeable.

Proposition 2.7 (Linearity, product, quotient, and chain rules). Suppose f, k are differentiable at $t \in (a, T)$ and $c_1, c_2 \in \mathbb{R}$. Then

$$ {}_a\mathcal{C}^{\alpha,\varphi}(c_1 f + c_2 k) = c_1\, {}_a\mathcal{C}^{\alpha,\varphi} f + c_2\, {}_a\mathcal{C}^{\alpha,\varphi} k, \tag{9} $$

$$ {}_a\mathcal{C}^{\alpha,\varphi}(fk) = f\, {}_a\mathcal{C}^{\alpha,\varphi} k + k\, {}_a\mathcal{C}^{\alpha,\varphi} f, \tag{10} $$

$$ {}_a\mathcal{C}^{\alpha,\varphi}(f/k) = \frac{k\, {}_a\mathcal{C}^{\alpha,\varphi} f - f\, {}_a\mathcal{C}^{\alpha,\varphi} k}{k^2} \quad (k(t) \neq 0). \tag{11} $$

If F is differentiable at $f(t)$, then

$$ {}_a\mathcal{C}^{\alpha,\varphi}(F \circ f)(t) = F'(f(t))\, {}_a\mathcal{C}^{\alpha,\varphi} f(t). \tag{12} $$

The formulas hold componentwise for vector-valued functions, and for compatible matrix-valued products the order of factors must be preserved.

Proof. According to Proposition 2.3, all the appropriate conformable derivatives can be calculated as the ordinary derivative divided by the same nonzero scalar $q(t)$. Ordinary differentiation yields that $(c_1 f + c_2 k)' = c_1 f' + c_2 k'$, and thus division by q establishes (9). In the case of the product, $(fk)' = f'k + fk'$, and hence division by q establishes (10). At the points where $k \neq 0$, the ordinary quotient rule implies $(f/k)' = (f'k - fk')/k^2$, and therefore division by q establishes (11). For compositions, we have $(F \circ f)' = F'(f)f'$ according to the ordinary chain rule, which implies (12). For matrix products, $(FG)' = F'G + FG'$; being scalar, q allows the same argument, but matrix multiplications cannot be commuted. In particular, these rules do not include the fractional Leibniz formula for nonlocal derivatives.

Proposition 2.8 (Exponential solution and variation of constants). Let $A \in \mathbb{R}^{n\times n}$ be constant, and let f be integrable with respect to the clock measure $q(t)\mathrm{d}t$. The unique clock-absolutelycontinuous solution of ${}_a\mathcal{C}^{\alpha,\varphi} x = Ax + f(t)$, $x(a) = x_a$, is

$$ x(t) = e^{Ah(t)} x_a + \int_a^t e^{A(h(t)-h(s))} f(s) q(s)\, \mathrm{d}s. \tag{13} $$

Proof. Write $y(\tau) = x(h^{-1}(\tau))$ and $\widehat{f}(\tau) = f(h^{-1}(\tau))$. The equation becomes $y' = Ay + \widehat{f}$ almost everywhere. Multiplying by $e^{-A\tau}$ and using the product rule yields $(e^{-A\tau} y(\tau))' = e^{-A\tau}\widehat{f}(\tau)$ because A commutes with its exponential. Integrate over $[0, \tau]$ and multiply by $e^{A\tau}$ to get $y(\tau) = e^{A\tau} x_a + \int_0^\tau e^{A(\tau-\sigma)}\widehat{f}(\sigma)\mathrm{d}\sigma$. The measure substitution gives (13). If two clock-AC solutions have the same initial state, their difference solves $z' = Az$ with $z(0) = 0$. Multiplication by $e^{-A\tau}$ makes the difference constant and equal to zero. Thus existence and uniqueness are established rather than assumed.

Proposition 2.9. If the admissible pairs (α_1, φ_1) and (α_2, φ_2) create the same clock function h on $[a, T]$, then the conformable derivatives will be equal at every point of differentiability in the interior; the integral measures will also be the same. Consequently, the same endpoint constraints and running cost measures produce the same optimal control and optimal value.

Proof. Equal clocks give equal derivatives $q_1 = q_2$ for $t > a$. Operators are given by f'/q_i and therefore have identical derivatives. In addition, the integration measures are identical and equal to $q_i \mathrm{d}t$. Application of variation-of-constants formula (13) will give the same result for a given control. Therefore, the sets of feasible functions and the cost functional are identical and have the same minimizers and infima. It also explains the reason why changing the nominal order α does not give a different observable evolution if φ changes accordingly.

Remark 2.10. The equation (5) is local in t. For a smooth trajectory, it lacks history-dependent convolution. Therefore, any paper which claims a nonlocal memory property from the above equation is mathematically wrong. However, a nonlinear clock can still make sense from the point of data fitting or prescribed time modulation.

3. Two distinct finit e-horizon optimal control models

Let $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, with constant matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and symmetric $Q \succeq 0, R \succ 0, G \succeq 0$. We consider

$$ {}_a\mathcal{C}^{\alpha,\varphi}x(t) = Ax(t) + Bu(t), \qquad x(a) = x_a. \tag{14}$$

For a fixed endpoint, impose $x(T) = x_T$ and omit the terminal penalty, or regard that penalty as constant. For a free endpoint, use the terminal quadratic penalty $\frac{1}{2}x(T)^{\mathsf{T}}Gx(T)$. The *natural-clock* functional is

$$J_h(u) = \frac{1}{2}\int_a^T (x^{\mathsf{T}}Qx + u^{\mathsf{T}}Ru)q(t)\mathrm{d}t + \frac{1}{2}x(T)^{\mathsf{T}}Gx(T), \tag{15}$$

while the physical-time functional is

$$J_{\text{phys}}(u) = \frac{1}{2}\int_a^T (x^{\mathsf{T}}Qx + u^{\mathsf{T}}Ru)\mathrm{d}t + \frac{1}{2}x(T)^{\mathsf{T}}Gx(T). \tag{16}$$

These are not interchangeable unless q is constant or a particular invariance applies.

Theorem 3.1 (Existence and uniqueness). Assume the prescribed fixed endpoint is reachable in operational time, if such a condition is imposed. Then the natural-clock problem (14)–(15) possesses a unique optimal control $u^* \in L^2([0,S];\mathbb{R}^m)$ in clock coordinates. For the physicaltime problem the same assertion holds if, in addition, q is bounded above and below by positive constants almost everywhere on $[a,T]$.

Proof. We give the direct-method argument in detail. In operational time set $H = L^2(0,S;\mathbb{R}^m)$ with inner product $\langle v_1, v_2\rangle_H = \int_0^S v_1(\tau)^{\mathsf{T}}v_2(\tau)\mathrm{d}\tau$. Proposition 2.8 gives the affine state map

$$y_v(\tau) = e^{A\tau}x_a + (\mathcal{L}v)(\tau), \qquad (\mathcal{L}v)(\tau) = \int_0^\tau e^{A(\tau-s)}Bv(s)\mathrm{d}s.$$

Let $M = \max_{0\le s\le S}\|e^{As}B\|$. Cauchy–Schwarz yields $\|(\mathcal{L}v)(\tau)\| \le M\sqrt{S}\,\|v\|_H$ for each τ, so $\mathcal{L}$ is bounded from H into $C([0,S];\mathbb{R}^n)$, hence also into $L^2(0,S;\mathbb{R}^n)$. The terminal map $Kv = (\mathcal{L}v)(S)$ is bounded and linear. A fixed-endpoint constraint is exactly the affine set $\mathcal{A} = \{v \in H : Kv = x_T - e^{AS}x_a\}$. It is nonempty by reachability and norm closed because K is continuous. It is also weakly closed: if $v_j \rightharpoonup v$ weakly in H, then the finite-dimensional vectors Kv_j converge to Kv.

Let $r > 0$ denote the smallest eigenvalue of R. Since Q, G are positive semidefinite,

$$J_h(v) \ge \frac{r}{2}\|v\|_H^2.$$

Choose a minimizing sequence in $\mathcal{A}$ (or in all H for the free terminal case). Its cost is bounded above, so the inequality makes the sequence bounded in the Hilbert space H. Passing to a subsequence gives weak convergence $v_j \rightharpoonup v^*$. The affine constraint remains satisfied. The terms $\| Q^{1/2}y_v \|_{L^2}^2$, $\| R^{1/2}v \|_H^2$, and $\| G^{1/2}y_v(S) \|^2$ are squares of norms of bounded affine images of v; each is weakly lower semicontinuous. Hence $J_h(v^*) \le \liminf_j J_h(v_j)$, proving existence. For distinct feasible v_1, v_2 and $0 < \theta < 1$, direct expansion of the quadratic control term gives

$$\theta\|R^{1/2}v_1\|_H^2 + (1-\theta)\|R^{1/2}v_2\|_H^2 - \|R^{1/2}(\theta v_1 + (1-\theta)v_2)\|_H^2$$
$$= \theta(1-\theta)\|R^{1/2}(v_1 - v_2)\|_H^2 > 0.$$

The state and terminal quadratic terms are convex because their weights are semidefinite. Thus J_h is strictly convex on the feasible affine set, proving uniqueness.

For physical time, the transformed weight w obeys $0 < c \le w \le C$. The weighted control term satisfies $\int_0^S wv^\top Rv\mathrm{d}\tau \ge cr\|v\|_H^2$. All weighted state maps remain bounded and the weighted quadratic forms are weakly lower semicontinuous because multiplication by $\sqrt{w}$ is bounded on L^2. Every preceding step therefore applies unchanged. Without the lower bound on w, coercivity on $L^2(0,S)$ is not automatic; Theorem 7.3 will supply an explicit example where attainment fails. □

4. Necessary and sufficient Hamiltonian conditions

We use the operational time τ because the weighted measures are then transparent. Define

$$w(\tau) = \frac{1}{q(h^{-1}(\tau))}, \qquad L_\rho(y,v,\tau) = \frac{\rho(\tau)}{2}(y^\top Qy + v^\top Rv), \tag{17}$$

where $\rho = 1$ represents the natural cost and $\rho = w$ the physical-time cost.

Theorem 4.1 (Optimality and transversality). Let $\rho\colon [0,S] \to (0,\infty)$ be measurable and bounded above and below by positive constants. If (y^*, v^*) minimizes

$$\frac{1}{2}\int_0^S \rho(\tau)(y^\top Qy + v^\top Rv)\mathrm{d}\tau + \frac{1}{2}y(S)^\top Gy(S), \quad y' = Ay + Bv, \quad y(0) = x_a, \tag{18}$$

then there exists an absolutely continuous costate p satisfying

$$y' = Ay + Bv, \tag{19}$$
$$p' = -\rho Qy - A^\top p, \tag{20}$$
$$v = -\rho^{-1}R^{-1}B^\top p. \tag{21}$$

If $y(S)$ is free, $p(S) = Gy(S)$; if $y(S) = x_T$ is prescribed, $p(S)$ is an unrestricted terminal multiplier. Conversely, any feasible triple satisfying these equations and the appropriate terminal condition is the unique global optimum.

Proof. Step 1: derivative of the state map. For an admissible perturbation $k \in L^2(0,S;\mathbb{R}^m)$, the perturbed state y_ϵ associated with $v + \epsilon k$ satisfies the linear system, so $y_\epsilon = y + \epsilon z$ exactly, where

$$z' = Az + Bk, \qquad z(0) = 0, \qquad z(\tau) = \int_0^\tau e^{A(\tau-s)}Bk(s)\,\mathrm{d}s. \tag{22}$$

For a free terminal state every k is allowed; for a prescribed endpoint only perturbations satisfying $z(S) = 0$ are feasible.

Step 2: the derivative of the objective. Expanding the squares and retaining the terms linear in ϵ gives

$$DJ(v)[k] = \int_0^S \rho(\tau)(y^\top Qz + v^\top Rk)\,\mathrm{d}\tau + y(S)^\top Gz(S). \tag{23}$$

The derivative is continuous in k, since the state map is bounded on L^2 and ρ is bounded.
Step 3: construction of a costate. First define p_0 as the unique absolutely continuous solution of

$$p_0' = -A^{\mathsf{T}} p_0 - \rho Q y, \qquad p_0(S) = G y(S).$$

Existence and uniqueness follow directly by reversing time and applying the linear variation-ofconstants formula. Integrating the product $(p_0^{\mathsf{T}} z)'$ using (22) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(p_0^{\mathsf{T}} z) &= (-A^{\mathsf{T}} p_0 - \rho Q y)^{\mathsf{T}} z + p_0^{\mathsf{T}}(Az + Bk) \\ &= -\rho y^{\mathsf{T}} Q z + p_0^{\mathsf{T}} B k.\end{aligned}$$

Since $z(0) = 0$ and $p_0(S) = Gy(S)$, formula (23) becomes

$$DJ(v)[k] = \int_0^S (\rho R v + B^{\mathsf{T}} p_0)^{\mathsf{T}} k \, \mathrm{d}\tau. \tag{24}$$

For a free terminal state, an optimum satisfies $DJ(v^*)[k] = 0$ for all $k \in L^2$, and the fundamental lemma for the L^2 inner product yields $\rho R v^* + B^{\mathsf{T}} p_0 = 0$ almost everywhere. The state, adjoint, stationarity, and free-endpoint conditions follow. For a fixed terminal state, define the bounded terminal-response map $Kk = z(S)$ from (22). At a constrained optimum, $DJ(v^*)$ vanishes on $\ker K$. Since K has finite-dimensional range, its adjoint has closed range and $(\ker K)^{\perp} = \mathrm{ran} K^*$. Consequently there exists $\mu \in \mathbb{R}^n$ such that $DJ(v^*)[k] + \mu^{\mathsf{T}} K k = 0$ for every k. Put

$$p(\tau) = p_0(\tau) + e^{A^{\mathsf{T}}(S-\tau)} \mu.$$

Then $p' = -A^{\mathsf{T}} p - \rho Q y$, $p(S) = Gy(S) + \mu$, and the identity $\mu^{\mathsf{T}} K k = \int_0^S (B^{\mathsf{T}} e^{A^{\mathsf{T}}(S-\tau)} \mu)^{\mathsf{T}} k \mathrm{d}\tau$ turns the preceding multiplier equation into $\rho R v + B^{\mathsf{T}} p = 0$ almost everywhere. No unjustified surjectivity assumption on the terminal map is needed; the multiplier need not be unique when the pair (A, B) is uncontrollable.

Step 4: sufficiency and global uniqueness. Let (y, v, p) satisfy the asserted system and terminal rule, and let $(\widetilde{y}, \widetilde{v})$ be any feasible competitor. Set $z = \widetilde{y} - y$ and $k = \widetilde{v} - v$, so $z' = Az + Bk$, $z(0) = 0$, and $z(S) = 0$ in the fixed-endpoint case. Exact expansion of the quadratic functional gives

$$J(\widetilde{v}) - J(v) = DJ(v)[k] + \frac{1}{2} \int_0^S \rho (z^{\mathsf{T}} Q z + k^{\mathsf{T}} R k) \, \mathrm{d}\tau + \frac{1}{2} z(S)^{\mathsf{T}} G z(S). \tag{25}$$

The same integration-by-parts calculation as in Step 3, now with the actual costate p, gives

$$DJ(v)[k] = \int_0^S (\rho R v + B^{\mathsf{T}} p)^{\mathsf{T}} k \mathrm{d}\tau + [Gy(S) - p(S)]^{\mathsf{T}} z(S) = 0.$$

The integral vanishes by stationarity. Its boundary term is zero either because $z(S) = 0$ or because $p(S) = Gy(S)$. Hence

$$J(\widetilde{v}) - J(v) \geq \frac{1}{2} \int_0^S \rho(\tau) k(\tau)^{\mathsf{T}} R k(\tau) \mathrm{d}\tau. \tag{26}$$

Because $\rho \geq c > 0$ and $R \succ 0$, the right-hand side is zero only if $k = 0$ almost everywhere. Thus the stationary feasible pair is the unique global minimizer. This proves sufficiency independently of any vague second-variation criterion.

Corollary 4.2 (Equations in physical time). For the natural cost, let $p(t)$ denote the costate composed with h. The optimality system becomes

$${}_a\mathcal{C}^{\alpha,\varphi} x = Ax + Bu, \quad {}_a\mathcal{C}^{\alpha,\varphi} p = -Qx - A^{\mathsf{T}} p, \quad u = -R^{-1} B^{\mathsf{T}} p. \tag{27}$$

For the physical-time cost, with $w(t) = 1/q(t)$, the correct system is

$$ {}_a\mathcal{C}^{\alpha,\varphi}x = Ax + Bu, \quad {}_a\mathcal{C}^{\alpha,\varphi}p = -w(t)Qx - A^{\mathsf{T}}p, \quad u = -\frac{1}{w(t)}R^{-1}B^{\mathsf{T}}p. \tag{28} $$

The natural-cost formula cannot generally be reused for (16).

Proof. For the natural functional, the operational-time Hamiltonian equations are $y' = Ay + Bv$, $p' = -Qy - A^{\mathsf{T}}p$, and $v = -R^{-1}B^{\mathsf{T}}p$. Compose state, costate, and control with $\tau = h(t)$. The time-change identity (6) converts both ordinary derivatives into conformable derivatives, proving (27). For physical time, $\rho(\tau) = w(\tau)$, so the same operations give ${}_a\mathcal{C}^{\alpha,\varphi}p = -wQx - A^{\mathsf{T}}p$ and $u = -w^{-1}R^{-1}B^{\mathsf{T}}p$. Here $w(t)$ abbreviates $w(h(t)) = 1/q(t)$. The factor w multiplying the state penalty and its reciprocal multiplying the control stationarity term arise from different derivatives of the weighted Hamiltonian and cannot be suppressed.

Theorem 4.3 (Convex pointwise control constraints, free terminal state). Let $U \subset \mathbb{R}^m$ be a nonempty closed convex set containing the origin. Suppose the terminal state is free, $v(\tau) \in U$ almost everywhere, and the assumptions of Theorem 4.1 hold. The unique optimal control has an adjoint satisfying the state and costate equations together with the variational inequality

$$ (\rho R v^* + B^{\mathsf{T}}p)^{\mathsf{T}}(z - v^*) \geq 0 \quad \textit{for all } z \in U \textit{ and a.e. } \tau. \tag{29} $$

Equivalently, with the R-metric projection Π_U^R, *it satisfies*

$$ v^*(\tau) = \Pi_U^R\left(-\rho(\tau)^{-1}R^{-1}B^{\mathsf{T}}p(\tau)\right). \tag{30} $$

Any feasible state-control-costate satisfying these conditions and $p(S) = Gy(S)$ is the unique global minimizer.

Proof. The set $\mathcal{U} = \{v \in L^2(0, S; \mathbb{R}^m) : v(\tau) \in U \text{ a.e.}\}$ is convex and closed in L^2. It is nonempty since $0 \in U$. The same coercivity and lower-semicontinuity proof as Theorem 3.1 gives a unique minimizer. For any competing $z \in \mathcal{U}$ and $0 \leq \theta \leq 1$, the control $(1-\theta)v^* + \theta z$ is admissible. Differentiating the cost at $\theta = 0^+$ and using (24) yields

$$ \int_0^S (\rho R v^* + B^{\mathsf{T}}p)^{\mathsf{T}}(z - v^*)\mathrm{d}\tau \geq 0. $$

Since the admissible control can be changed on an arbitrary measurable time subset and U is convex, the integral inequality implies the pointwise inequality (29); one can first apply it to a countable dense subset of U and then use continuity in z to obtain all of U simultaneously.

The R-metric projection $\Pi_U^R(b)$ is the unique minimizer of $z \mapsto \frac{1}{2}(z-b)^{\mathsf{T}}R(z-b)$ over U, because $R \succ 0$. Expanding this quadratic along line segments shows that $v = \Pi_U^R(b)$ if and only if $(R(v-b))^{\mathsf{T}}(z-v) \geq 0$ for all $z \in U$. Choosing $b = -\rho^{-1}R^{-1}B^{\mathsf{T}}p$ gives (29) after multiplication by the positive scalar ρ, hence proves (30). For sufficiency, the cost difference (25) has linear term $\int_0^S (\rho R v^* + B^{\mathsf{T}}p)^{\mathsf{T}}(\tilde{v} - v^*)\mathrm{d}\tau \geq 0$ and nonnegative quadratic remainder. Strict convexity of the control term again proves uniqueness. The pointwise projection does not, in general, give a matrix-exponential closed form. □

5. Closed forms for autonomous natural-clock LQ problems

For $\rho = 1$, define the constant Hamiltonian matrix

$$ \mathcal{M} = \begin{pmatrix} A & -BR^{-1}B^{\mathsf{T}} \\ -Q & -A^{\mathsf{T}} \end{pmatrix}, \qquad \begin{pmatrix} y(\tau) \\ p(\tau) \end{pmatrix} = e^{\mathcal{M}\tau}\begin{pmatrix} x_a \\ p_0 \end{pmatrix}. \tag{31} $$

Write $e^{\mathcal{M}S} = \left(\begin{smallmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{smallmatrix}\right)$ in conforming $n \times n$ blocks.

Theorem 5.1 (Fixed and free endpoints). For a prescribed endpoint $y(S) = x_T$, if F_{12} is invertible, then

$$p_0 = F_{12}^{-1}(x_T - F_{11}x_a), \tag{32}$$

with y, p, v recovered from (31) and $v = -R^{-1}B^{\mathsf{T}}p$. If the endpoint is free with penalty G, and $F_{22} - GF_{12}$ is invertible, then

$$p_0 = (F_{22} - GF_{12})^{-1}(GF_{11} - F_{21})x_a. \tag{33}$$

In particular these formulas hold whenever their indicated matrices are invertible; controllability of (A, B) guarantees invertibility of F_{12} for $Q \succeq 0, R \succ 0$.

Proof. Construction of the propagator. Combining the state, adjoint, and stationarity equations with $\rho = 1$ gives the constant-coefficient system $z' = \mathcal{M}z$ for $z = (y^{\mathsf{T}}, p^{\mathsf{T}})^{\mathsf{T}}$. The absolutely and uniformly convergent matrix exponential series $e^{\mathcal{M}\tau} = \sum_{j=0}^{\infty} \mathcal{M}^j \tau^j / j!$ can be differentiated term by term on the compact time interval; hence its derivative is $\mathcal{M}e^{\mathcal{M}\tau}$ and its initial value is I_{2n}. Uniqueness of the linear system proves (31). Partitioning its terminal propagator yields

$$\begin{pmatrix} y(S) \\ p(S) \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} x_a \\ p_0 \end{pmatrix}.$$

For the prescribed state $y(S) = x_T$, rearrangement gives $F_{12}p_0 = x_T - F_{11}x_a$, which is (32) if F_{12} is nonsingular. For the free endpoint, insert $p(S) = Gy(S)$ into the two block rows and rearrange to obtain $(F_{22} - GF_{12})p_0 = (GF_{11} - F_{21})x_a$, which gives (33) under the indicated invertibility hypothesis. Why controllability makes F_{12} nonsingular. Suppose $F_{12}\zeta = 0$ for some $\zeta \in \mathbb{R}^n$. Take the canonical solution with $y(0) = 0$ and $p(0) = \zeta$. Its terminal state is $y(S) = 0$. The corresponding $v = -R^{-1}B^{\mathsf{T}}p$ satisfies the necessary conditions for the zero-to-zero fixed-endpoint problem and, by Theorem 4.1, is a global minimizer. The identically zero state and control are feasible and have zero cost, so strict convexity forces $v = 0$ almost everywhere. The state must then be $y = 0$. Stationarity gives $B^{\mathsf{T}}p = 0$, and the adjoint reduces to $p' = -A^{\mathsf{T}}p$, hence

$$B^{\mathsf{T}}e^{-A^{\mathsf{T}}\tau}\zeta = 0 \qquad (0 \le \tau \le S).$$

Repeated differentiation with respect to τ at zero gives $B^{\mathsf{T}}(A^{\mathsf{T}})^k\zeta = 0$ for $k = 0,1, \dots, n-1$. Equivalently, ζ is orthogonal to the columns of the Kalman matrix $[B, AB, \dots, A^{n-1}B]$. Controllability says that this matrix has rank n, so $\zeta = 0$. Thus $\ker F_{12} = \{0\}$ and F_{12} is invertible. Invertibility of the $F_{22} - GF_{12}$ matrix under controllability follows from the same reasoning: if the matrix nullifies ζ, then the canonical trajectory $y(0) = 0$, $p(0) = \zeta$ will satisfy the free terminal condition $p(S) = Gy(S)$. Sufficient condition entails that its control will be the unique zero-control optimum under zero initial conditions; hence $y = 0$, $B^{\mathsf{T}}p = 0$, and $\zeta = 0$ follows by the Kalman reasoning. This information is not necessary for the conditional formulas but serves as an additional sufficient condition for both.

Proposition 5.2 (Exact value identity). For any stationary natural-clock LQ trajectory satisfying (27),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p^{\mathsf{T}}y) = -y^{\mathsf{T}}Qy - v^{\mathsf{T}}Rv, \qquad J_h^* = \frac{1}{2}[p(0)^{\mathsf{T}}x_a - p(S)^{\mathsf{T}}y(S) + y(S)^{\mathsf{T}}Gy(S)]. \tag{34}$$

For a free endpoint $p(S) = Gy(S)$, this simplifies to $J_h^* = \frac{1}{2}p(0)^{\mathsf{T}}x_a$. It also simplifies for $y(S) = 0$ and $G = 0$.

Proof. The absolutely continuous functions p and y satisfy the product rule almost everywhere:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}(p^{\mathsf{T}}y) &= (p')^{\mathsf{T}}y + p^{\mathsf{T}}y' \\ &= (-Qy - A^{\mathsf{T}}p)^{\mathsf{T}}y + p^{\mathsf{T}}(Ay + Bv) \\ &= -y^{\mathsf{T}}Qy - p^{\mathsf{T}}Ay + p^{\mathsf{T}}Ay + p^{\mathsf{T}}Bv \\ &= -y^{\mathsf{T}}Qy - v^{\mathsf{T}}Rv.\end{aligned}$$

The last equality is the stationarity identity $B^{\mathsf{T}}p = -Rv$. Since $p^{\mathsf{T}}y$ is absolutely continuous, integrating from 0 to S gives $\int_0^S (y^{\mathsf{T}}Qy + v^{\mathsf{T}}Rv)\mathrm{d}\tau = p(0)^{\mathsf{T}}x_a - p(S)^{\mathsf{T}}y(S)$. Add the terminal penalty $y(S)^{\mathsf{T}}Gy(S)/2$ to obtain (34). For the free endpoint, substitute $p(S) = Gy(S)$: its two terminal terms cancel. For prescribed $y(S) = 0$, both terminal terms vanish if $G = 0$. The identity is valid on any stationary solution; its notation J_h^* refers to the optimal one whose existence is established separately. □

6. Physical-time Riccati equation and comparison principle

The physical-time functional is transformed to (18) with $\rho = w(\tau)$. Even with constant A, B, Q, R, w is generally time dependent.

Theorem 6.1 (Nonautonomous Riccati feedback)**.** *Assume $0 < c \le w(\tau) \le C < \infty$. For a free endpoint and terminal penalty $G \succeq 0$, the finite-horizon value function is $V(\tau, y) = \frac{1}{2}y^{\mathsf{T}}P(\tau)y$, where P is the unique symmetric nonnegative solution on $[0, S]$ of*

$$-P' = wQ + A^{\mathsf{T}}P + PA - \frac{1}{w}PBR^{-1}B^{\mathsf{T}}P, \qquad P(S) = G. \tag{35}$$

The optimal feedback is

$$v^*(\tau) = -\frac{1}{w(\tau)}R^{-1}B^{\mathsf{T}}P(\tau)y^*(\tau), \qquad J^*_{\mathrm{phys}} = \frac{1}{2}x_a^{\mathsf{T}}P(0)x_a. \tag{36}$$

Proof. We construct the solution and prove its properties without assuming an HJB equation in advance. Put $D(\tau) = w(\tau)^{-1}BR^{-1}B^{\mathsf{T}}$, which is symmetric and positive semidefinite, and consider the $2n \times n$ matrix functions Y, Z defined by the terminal-value linear system

$$Y' = AY - DZ, \qquad Z' = -wQY - A^{\mathsf{T}}Z, \qquad Y(S) = I_n, \quad Z(S) = G. \tag{37}$$

Since w and $1/w$ are bounded and measurable, this linear system has a unique absolutely continuous solution on the entire compact interval. Step 1: invertibility of $Y(\tau)$. Fix $\tau_0 \in [0, S]$ and assume $Y(\tau_0)\zeta = 0$. The vector functions $y = Y\zeta$, $p = Z\zeta$ satisfy the canonical equations on $[\tau_0, S]$, with $y(\tau_0) = 0$, $p(S) = Gy(S)$, and $v = -w^{-1}R^{-1}B^{\mathsf{T}}p$. Direct differentiation gives the weighted energy identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p^{\mathsf{T}}y) = -w(y^{\mathsf{T}}Qy + v^{\mathsf{T}}Rv). \tag{38}$$

Integration from τ_0 to S and the two endpoint conditions yield

$$0 = y(S)^{\mathsf{T}}Gy(S) + \int_{\tau_0}^{S} w(y^{\mathsf{T}}Qy + v^{\mathsf{T}}Rv)\mathrm{d}\tau.$$

All terms are nonnegative. In particular $v = 0$ almost everywhere because $w > 0$ and $R \succ 0$. With $y(\tau_0) = 0$, uniqueness of $y' = Ay + Bv$ gives $y = 0$, hence $y(S) = Y(S)\zeta = \zeta = 0$. Therefore $Y(\tau_0)$ is injective and invertible for every τ_0. Step 2: derivation of Riccati. Define $P(\tau) = Z(\tau)Y(\tau)^{-1}$. It is absolutely continuous because Y and Z are absolutely continuous and

Y^{-1} is continuous and bounded on $[0,S]$. The derivative of an inverse matrix is $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$ almost everywhere. Substituting (37) gives

$$\begin{aligned} P' &= (Z' - PY')Y^{-1} \\ &= (-wQY - A^{\mathsf{T}}Z - PAY + PDZ)Y^{-1} \\ &= -wQ - A^{\mathsf{T}}P - PA + PDP. \end{aligned}$$

Since $P(S) = ZI^{-1} = G$, this is precisely (35). Step 3: symmetry and nonnegativity. Differentiate $Y^{\mathsf{T}}Z - Z^{\mathsf{T}}Y$ using (37). Terms involving A cancel in pairs, as do terms involving the symmetric matrices D and Q, so $(Y^{\mathsf{T}}Z - Z^{\mathsf{T}}Y)' = 0$ almost everywhere. At S the quantity is $G - G^{\mathsf{T}} = 0$; hence $Y^{\mathsf{T}}Z = Z^{\mathsf{T}}Y$ for every τ, and multiplication by $Y^{-\mathsf{T}}$ and Y^{-1} yields $P = P^{\mathsf{T}}$. For any τ_0 and any initial vector η, choose $\zeta = Y(\tau_0)^{-1}\eta$. The corresponding canonical trajectory satisfies $y(\tau_0) = \eta$, $p(\tau_0) = P(\tau_0)\eta$. Integrating (38) gives

$$\eta^{\mathsf{T}}P(\tau_0)\eta = \int_{\tau_0}^{S} w(y^{\mathsf{T}}Qy + v^{\mathsf{T}}Rv)\mathrm{d}\tau + y(S)^{\mathsf{T}}Gy(S) \geq 0.$$

Thus $P(\tau) \succeq 0$ for all τ. Step 4: verification of the optimal feedback. Let y solve the closed-loop equation $y' = (A - DP)y$ with $y(0) = x_a$ and put $v^* = -w^{-1}R^{-1}B^{\mathsf{T}}Py$. For an arbitrary admissible candidate v with its own state z, differentiate $z^{\mathsf{T}}Pz$ and use the Riccati equation. Algebraic completion of squares yields the exact identity

$$\begin{aligned} &\frac{1}{2}\int_0^S w(z^{\mathsf{T}}Qz + v^{\mathsf{T}}Rv)\mathrm{d}\tau + \frac{1}{2}z(S)^{\mathsf{T}}Gz(S) \\ &\quad = \frac{1}{2}x_a^{\mathsf{T}}P(0)x_a + \frac{1}{2}\int_0^S w(v + w^{-1}R^{-1}B^{\mathsf{T}}Pz)^{\mathsf{T}}R(v + w^{-1}R^{-1}B^{\mathsf{T}}Pz)\mathrm{d}\tau. \end{aligned} \tag{39}$$

The nonnegative integral vanishes exactly for the stated feedback evaluated along z. The resulting closed-loop initial-value problem has a unique solution, so v^* is the unique global optimizer. The initial term in (39) proves the value formula. Finally, uniqueness of the symmetric Riccati solution on $[0,S]$ follows from uniqueness of the initial-value problem after reversing time, or directly by applying (39) to any competing nonnegative solution. This completes existence, symmetry, positivity, feedback optimality, and uniqueness in one argument.

Corollary 6.2 (Physical-time Riccati law in the original variable)**.** *Set $\widehat{P}(t) = P(h(t))$, where P solves (35). If q is bounded above and below by positive constants, then $\widehat{P}$ is absolutely continuous and satisfies almost everywhere*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\widehat{P} = Q + q(t)(A^{\mathsf{T}}\widehat{P} + \widehat{P}A) - q(t)^2\widehat{P}BR^{-1}B^{\mathsf{T}}\widehat{P}, \qquad \widehat{P}(T) = G. \tag{40}$$

The physical-time optimal feedback and state equation are

$$u^*(t) = -q(t)R^{-1}B^{\mathsf{T}}\widehat{P}(t)x^*(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}x^*(t) = q(t)\big(Ax^*(t) + Bu^*(t)\big). \tag{41}$$

Proof. The ordinary chain rule for absolutely continuous functions gives $\widehat{P}'(t) = q(t)P'(h(t))$ almost everywhere. Substitute the right side of (35) for $-P'$ and use $w(h(t)) = 1/q(t)$ to obtain $-\widehat{P}' = q[(1/q)Q + A^{\mathsf{T}}\widehat{P} + \widehat{P}A - q\widehat{P}BR^{-1}B^{\mathsf{T}}\widehat{P}]$, which simplifies to (40). Similarly (36) becomes $u = -qR^{-1}B^{\mathsf{T}}\widehat{P}x$. Finally, the local differential representation ${}_a\mathcal{C}^{\alpha,\varphi}x = x'/q$ gives the state equation (41). Notice the squared factor q^2 in the quadratic Riccati term; omitting it would correspond to the wrong cost measure. □

Theorem 6.3 (Measure comparison). Suppose $0 < c \le w \le C$ and $G = 0$, and the same dynamics and endpoint constraints are imposed for both measures. Then their optimal values satisfy

$$cV_h \le V_{\text{phys}} \le CV_h. \tag{42}$$

The assertion also holds for $G \neq 0$ after replacing c by $\min(c, 1)$ and C by $\max(C, 1)$.

Proof. Let $\mathcal{A}$ be the common admissible set in operational time. Define the nonnegative integrand

$$F_v(\tau) = \tfrac{1}{2}(y_v(\tau)^{\mathsf{T}} Q y_v(\tau) + v(\tau)^{\mathsf{T}} R v(\tau)) \ge 0.$$

For each feasible v, the pointwise inequalities $cF_v \le wF_v \le CF_v$ hold almost everywhere. Integration gives $cJ_h(v) \le J_{\text{phys}}(v) \le CJ_h(v)$ when $G = 0$. For the lower bound, for every $v \in \mathcal{A}$, $J_{\text{phys}}(v) \ge cJ_h(v) \ge cV_h$, hence taking the infimum over v yields $V_{\text{phys}} \ge cV_h$. For the upper bound, take the unique natural-clock minimizer v_h^*. Since it belongs to the same feasible set, $V_{\text{phys}} \le J_{\text{phys}}(v_h^*) \le CJ_h(v_h^*) = CV_h$. If $G \succeq 0$ is not zero, its terminal contribution $E_v = \frac{1}{2} y_v(S)^{\mathsf{T}} G y_v(S)$ is common to both costs and nonnegative. Put $F_0 = \int_0^S F_v \mathrm{d}\tau$ and $F_w = \int_0^S wF_v \mathrm{d}\tau$. The elementary bounds

$$\min(c, 1)(F_0 + E_v) \le F_w + E_v \le \max(C, 1)(F_0 + E_v)$$

replace the previous inequalities and give the announced constants through the same infimum argument.

Remark 6.4 (Interpretation and potential novelty). The closed form (31) for natural-clock objectives is a time-transformed version of classical LQ control. It should not be presented as a wholly new class of exponential solutions. A useful focus of the physical-time formulation is the explicit relation between clock choice and weighting, including (35) and (42); the broader priority question must still be assessed against weighted-LQ literature before a journal-submission novelty claim.

7. A physical-time closed form: minimum-energy steering

There is a useful physical-time problem in which the optimal control remains explicit even when the general Riccati equation is nonautonomous. Let $Q = G = 0$, prescribe $x(a) = x_a$, $x(T) = x_T$, and minimize

$$J_E(u) = \frac{1}{2} \int_a^T u(t)^{\mathsf{T}} R u(t) \, \mathrm{d}t \tag{43}$$

under ${}_a\mathcal{C}^{\alpha,\varphi} x = Ax + Bu$. Here the cost of an actuator is measured by actual elapsed time, not by the φ-clock.

Theorem 7.1 (Weighted Gramian and explicit optimal actuator). Assume (A, B) is controllable, $q(t) > 0$ for $t > a$, and $q \in L^2(a, T)$. Set $S = h(T)$, $\xi = x_T - e^{AS} x_a$, and

$$W_q = \int_a^T q(t)^2 e^{A(S-h(t))} B R^{-1} B^{\mathsf{T}} e^{A^{\mathsf{T}}(S-h(t))} \, \mathrm{d}t. \tag{44}$$

Then W_q is symmetric positive definite, the physical-time minimum-energy problem has a unique optimal control in $L^2(a, T; \mathbb{R}^m)$, and

$$u^*(t) = q(t) R^{-1} B^{\mathsf{T}} e^{A^{\mathsf{T}}(S-h(t))} W_q^{-1} \xi, \tag{45}$$

$$J_E(u^*) = \frac{1}{2} \xi^{\mathsf{T}} W_q^{-1} \xi. \tag{46}$$

Proof. Step 1: terminal constraint as a bounded linear operator. Proposition 2.8 with $f = Bu$ gives

$$x(T) = e^{AS}x_a + \int_a^T q(t)e^{A(S-h(t))}Bu(t)\mathrm{d}t.$$

Thus the target requirement is $Ku = \xi$, where $(Ku) = \int_a^T q(t)e^{A(S-h(t))}Bu(t)\mathrm{d}t$. Let $C_A = \max_{0\le\tau\le S}\|e^{A\tau}B\|$. By Cauchy–Schwarz, $\|Ku\| \le C_A\|q\|_{L^2(a,T)}\|u\|_{L^2(a,T)}$. Hence K is a bounded map from the physical-time control Hilbert space to $\mathbb{R}^n$. This is exactly where the assumption $q \in L^2$ is needed.

Step 2: weighted adjoint and Gramian. Endow the control space with $\langle u, v\rangle_R = \int_a^T u^{\mathsf{T}}Rv\mathrm{d}t$. For every u and $z \in \mathbb{R}^n$,

$$\begin{aligned} z^{\mathsf{T}}Ku &= \int_a^T u(t)^{\mathsf{T}}\left[q(t)B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h(t))}z\right]\mathrm{d}t \\ &= \left\langle u,\ q(\cdot)R^{-1}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h(\cdot))}z\right\rangle_R. \end{aligned}$$

Therefore $K^*z = qR^{-1}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h)}z$, and multiplication of K by this adjoint yields $KK^* = W_q$ as defined in (44). In particular W_q is symmetric and nonnegative. Step 3: strict positivity. Suppose $z^{\mathsf{T}}W_qz = 0$. Then

$$\int_a^T q(t)^2\|R^{-1/2}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h(t))}z\|^2\mathrm{d}t = 0.$$

Because $q(t) > 0$ throughout $(a, T]$, the continuous function $B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h(t))}z$ vanishes almost everywhere and therefore everywhere on $(a, T]$. Since h maps this interval onto $(0, S]$, we have $B^{\mathsf{T}}e^{A^{\mathsf{T}}s}z = 0$ for every $s \in [0, S]$. Differentiating at $s = 0$ gives $B^{\mathsf{T}}(A^{\mathsf{T}})^kz = 0$ for $k = 0,1,\dots,n-1$, so controllability forces $z = 0$. Thus $W_q \succ 0$. Step 4: orthogonal projection and exact minimum. Define $u^* = K^*W_q^{-1}\xi$. Then $Ku^* = KK^*W_q^{-1}\xi = \xi$, so u^* reaches the target. Every other feasible control can be written $u = u^* + v$ with $Kv = 0$. The adjoint identity implies

$$\langle u^*, v\rangle_R = \langle K^*W_q^{-1}\xi, v\rangle_R = (W_q^{-1}\xi)^{\mathsf{T}}Kv = 0.$$

Pythagoras in the R-weighted Hilbert space gives $\| u \|_R^2 = \| u^* \|_R^2 + \| v \|_R^2 \ge \| u^* \|_R^2$, with equality only if $v = 0$. Finally, $\| u^* \|_R^2 = (W_q^{-1}\xi)^{\mathsf{T}}KK^*W_q^{-1}\xi = \xi^{\mathsf{T}}W_q^{-1}\xi$. Division by two yields (46). The derivation proves feasibility, minimality, uniqueness, and the value identity rather than relying on a guessed control formula.

Corollary 7.2 (Natural-clock minimum-energy law). For the matrices and endpoint constraints of Theorem 7.1, replace the physical-time cost by

$$J_h(u) = \tfrac{1}{2}\int_a^T u(t)^{\mathsf{T}}Ru(t)q(t)\mathrm{d}t.$$

If (A, B) is controllable, define

$$W_h = \int_0^S e^{A(S-\tau)}BR^{-1}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-\tau)}\mathrm{d}\tau.$$

Then $W_h \succ 0$ and $u_h^*(t) = R^{-1}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-h(t))}W_h^{-1}\xi$, with $J_h(u_h^*) = \frac{1}{2}\xi^{\mathsf{T}}W_h^{-1}\xi$.

Proof. In operational time the endpoint map is $\widetilde{K}v = \int_0^S e^{A(S-\tau)}Bv(\tau)\mathrm{d}\tau$. The adjoint in the R-weighted clock-time L^2 space is $\widetilde{K}^*z = R^{-1}B^{\mathsf{T}}e^{A^{\mathsf{T}}(S-\tau)}z$. Thus $\widetilde{K}\widetilde{K}^* = W_h$. Positivity follows by the same analytic/Kalman-rank argument as Step 3 of Theorem 7.1, now with $q = 1$ in clock time. The projection theorem in that proof produces $v_h^* = \widetilde{K}^*W_h^{-1}\xi$ and the value $\frac{1}{2}\xi^{\mathsf{T}}W_h^{-1}\xi$. Composing with $\tau = h(t)$ yields the stated physical-time expression. This distinction makes clear why the physical-time Gramian contains $q(t)^2\mathrm{d}t$ whereas the natural-clock Gramian contains only $q(t)\mathrm{d}t$ after substitution. □

Theorem 7.3 (A sharp singular-clock threshold)**.** *Consider $a = 0$, $T = 1$, $A = 0$, $B = R = 1$, $x(0) = 0$, $x(1) = d \neq 0$, $\varphi(t) = t^m$ with $m > 0$, and $0 < \alpha \leq 1$. Let $\beta = m\alpha$. For the physical-time objective $J_E(u) = \frac{1}{2}\int_0^1 u(t)^2\mathrm{d}t$:*

(i) If $\beta > \frac{1}{2}$, the unique minimizer is

$$u^*(t) = \frac{d(2\beta-1)}{m}\,t^{\beta-1}, \qquad J_E(u^*) = \frac{d^2(2\beta-1)}{2m^2}. \tag{47}$$

(ii) If $0 < \beta \leq \frac{1}{2}$, the infimum is zero but is not attained by any admissible control.

Admissibility in this theorem means $u \in L^2(0,1)$, $qu \in L^1(0,1)$, $x(t) = \int_0^t q(s)u(s)\mathrm{d}s$ (hence x is absolutely continuous), and the two endpoint conditions hold. The restriction $qu \in L^1$ is essential when $qotinL^2$.

Proof. For $\varphi(t) = t^m$ and $\beta = m\alpha$, differentiation gives

$$q(t) = \varphi'(t)[\varphi(t) - \varphi(0)]^{\alpha-1} = mt^{m-1}\,t^{m(\alpha-1)} = mt^{\beta-1}.$$

Since $\beta > 0$, this density belongs to $L^1(0,1)$ and $\int_0^1 q(t)\mathrm{d}t = m/\beta = 1/\alpha$, in agreement with the clock formula. The state constraint $x' = qu$ and $x(0) = 0$ implies

$$\int_0^1 q(t)u(t)\mathrm{d}t = d. \tag{48}$$

We separate the two parameter regimes.

Case $\beta > 1/2$. Now $\|q\|_{L^2}^2 = m^2\int_0^1 t^{2\beta-2}\mathrm{d}t = m^2/(2\beta-1) < \infty$. Cauchy–Schwarz applied to (48) gives $|d| \leq \|q\|_2\|u\|_2$, hence $J_E(u) = \|u\|_2^2/2 \geq d^2/(2\|q\|_2^2)$. Equality in Cauchy–Schwarz holds precisely when u is a scalar multiple of q almost everywhere. Imposing the moment constraint determines this scalar uniquely:

$$u^*(t) = \frac{d}{\|q\|_2^2}q(t) = \frac{d(2\beta-1)}{m}t^{\beta-1}.$$

Because q^2 is integrable, $qu^* \in L^1$; the state $x^*(t) = \int_0^t q(s)u^*(s)\mathrm{d}s$ is absolutely continuous and satisfies both endpoints. Substituting the L^2 norm gives (47), and the equality case of Cauchy–Schwarz proves uniqueness.

Case $0 < \beta \leq 1/2$. Here $q \in L^1 \setminus L^2$ because $2\beta - 2 \leq -1$. For each $0 < \epsilon < 1$, define

$$H_\epsilon = \int_\epsilon^1 q(s)^2\mathrm{d}s \in (0,\infty), \qquad u_\epsilon(t) = \frac{d}{H_\epsilon}q(t)\mathbf{1}_{[\epsilon,1]}(t).$$

Each u_ϵ belongs to L^2 because it vanishes near zero; moreover qu_ϵ belongs to L^1. The endpoint moment is exactly $\int_0^1 qu_\epsilon = dH_\epsilon/H_\epsilon = d$. Thus $x_\epsilon(t) = \int_0^t q(s)u_\epsilon(s)\mathrm{d}s$ is absolutely continuous and reaches d. The cost is

$$J_E(u_\epsilon) = \frac{1}{2}\frac{d^2}{H_\epsilon^2}\int_\epsilon^1 q(t)^2\mathrm{d}t = \frac{d^2}{2H_\epsilon}.$$

If $\beta < 1/2$, then $H_\epsilon = \frac{m^2}{1-2\beta}(\epsilon^{2\beta-1} - 1) \to \infty$; if $\beta = 1/2$, then $H_\epsilon = m^2\log(1/\epsilon) \to \infty$. Hence $J_E(u_\epsilon) \to 0$. Since every cost is nonnegative, the infimum is exactly zero. But cost zero forces $u = 0$ almost everywhere, and then $\int qu = 0$, contradicting $d \neq 0$. Therefore the infimum is not attained.

The explanation for the argument lies in the fact that when q does not belong to L^2, the functional $u \to \int qu$ is no longer continuous in the full physical-time L^2 space. A control sequence can satisfy the endpoint condition while its L^2 norm goes to zero.

Remark 7.4. In the case of $\varphi(t) = t$ the threshold is exactly $\alpha > 1/2$. Hence, it is possible that the physical time optimal control problem does not have a minimizer at $\alpha \leq 1/2$ despite the state equation being well-defined for the given admissible controls used in the proof. It is because of this that the assumption of uniform boundedness in Theorem 3.1 cannot be dropped easily.

8. Two exact scalar endpoint examples

Take $a = 0$, $T = 1$, $A = -1$, $B = 1$, $Q = R = 1$, $x(0) = 1$, $G = 0$ and the natural-clock objective. The canonical matrix satisfies

$$\mathcal{M} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \mathcal{M}^2 = 2I_2, \qquad e^{\mathcal{M}\tau} = I_2\cosh(\sqrt{2}\tau) + \frac{\mathcal{M}}{\sqrt{2}}\sinh(\sqrt{2}\tau). \tag{49}$$

Write $\nu = \sqrt{2}$, $S = h(1)$, and let $\tau = h(t)$. These exact formulas provide independent checks of numerical implementations and of the original examples in [1].

8.1 Example I: fixed final state $x(1) = 0$

Solving the endpoint equations gives

$$x^*(t) = \frac{\sinh[\nu(S - h(t))]}{\sinh(\nu S)}, \tag{50}$$

$$u^*(t) = \frac{\sinh[\nu(S - h(t))] - \nu\cosh[\nu(S - h(t))]}{\sinh(\nu S)}, \tag{51}$$

$$J_h^* = \frac{1}{2}(\nu\coth(\nu S) - 1). \tag{52}$$

Derivation and verification. The canonical equations with $A = -1, B = Q = R = 1$ are $y' = -y - p$, $p' = -y + p$, and $v = -p$. Eliminating $p = -y' - y$ from the first equation gives

$$y'' = -y' - p' = -y' + y - p = -y' + y + (y' + y) = 2y.$$

The general solution of $y'' = \nu^2 y$ is a linear combination of $e^{\nu\tau}$ and $e^{-\nu\tau}$. Writing it in the basis centered at S gives $y(\tau) = c_1\cosh(\nu(S - \tau)) + c_2\sinh(\nu(S - \tau))$. The final constraint $y(S) = 0$ forces $c_1 = 0$. The initial condition $y(0) = 1$ gives $c_2 = 1/\sinh(\nu S)$, which is (50). Since $v = y' + y$ from the state equation,

$$v(\tau) = \frac{-\nu\cosh(\nu(S-\tau)) + \sinh(\nu(S-\tau))}{\sinh(\nu S)},$$

which yields (51) after replacing τ by $h(t)$. Define $p = -v$. Then $p' = -y + p$ holds by substitution because $y'' = 2y$, so all canonical equations and both endpoint constraints are verified. The initial costate is $p(0) = \nu\coth(\nu S) - 1$. Since $y(S) = 0$ and $G = 0$, the exact energy identity gives $J_h^* = p(0)y(0)/2 = p(0)/2$, proving (52). The strict sufficiency theorem establishes that these stationary formulas give the unique global optimum.

8.2 Example II: free final state

Now impose $p(S) = 0$. Set $D_S = \cosh(\nu S) + \nu^{-1}\sinh(\nu S)$. The exact solution reads

$$x^*(t) = \frac{\cosh[\nu(S-h(t))] + \nu^{-1}\sinh[\nu(S-h(t))]}{D_S}, \tag{53}$$

$$u^*(t) = -\frac{\sinh[\nu(S-h(t))]}{\nu D_S}, \tag{54}$$

$$J_h^* = \frac{\sinh(\nu S)}{2\nu D_S}. \tag{55}$$

Derivation and verification. Again $y'' = \nu^2 y$ and $p = -y' - y$. The free endpoint transversality condition $p(S) = 0$ is therefore the Robin condition $y'(S) + y(S) = 0$. For $y(\tau) = c_1\cosh(\nu(S-\tau)) + c_2\sinh(\nu(S-\tau))$, we have $y(S) = c_1$ and $y'(S) = -\nu c_2$. Hence $-\nu c_2 + c_1 = 0$, or $c_2 = c_1/\nu$. The initial condition gives $c_1 = [\cosh(\nu S) + \nu^{-1}\sinh(\nu S)]^{-1} = D_S^{-1}$. This proves (53). Next,

$$\begin{aligned} v(\tau) &= y'(\tau) + y(\tau) \\ &= \frac{(\nu^{-1} - \nu)\sinh(\nu(S-\tau))}{D_S} = -\frac{\sinh(\nu(S-\tau))}{\nu D_S}, \end{aligned}$$

where the last equality uses $\nu^2 = 2$. This proves (54) and shows $v(S) = 0$, i.e. $p(S) = 0$. The initial costate is $p(0) = -v(0) = \sinh(\nu S)/(\nu D_S)$. By Proposition 5.2 the terminal contribution cancels, so $J_h^* = p(0)/2$ and (55) follows. The stationary pair is the unique global optimum by Theorem 4.1. For $\varphi(t) = t$ and $a = 0$, the clock is $h(t) = t^\alpha/\alpha$ and $S = 1/\alpha$. Table 1 is computed from the exact formulas, not estimated by reading an existing figure. The numerical values were also independently recomputed from the closed-form expressions and agree with the MATLAB output to better than 4×10^{-16} in absolute error.

Table 1: Exact minimal natural-clock costs for the scalar examples ($\varphi(t) = t$); displayed decimals are rounded.

α	0.50	0.70	0.85	1.00
Fixed final state	0.21206464	0.23242342	0.25973790	0.29594583
Free final state	0.20625963	0.20285236	0.19845398	0.19290930

9. Physical-time example: a clock-dependent control law

We now keep $A = -1$, $B = 1$, $Q = R = 1$, $x(0) = 1$ but minimize the physical-time cost

$$J_{\text{phys}}(u) = \frac{1}{2}\int_0^1 (x(t)^2 + u(t)^2)\mathrm{d}t. \tag{56}$$

Fix any $0 < \alpha \leq 1$ and construct the admissible reference function

$$\varphi_\eta(t) = \left[\alpha(t+\eta t^2)\right]^{1/\alpha}, \quad \eta \geq 0, \qquad h_\eta(t) = t + \eta t^2, \quad q_\eta(t) = 1 + 2\eta t. \tag{57}$$

Indeed, by construction $[\varphi_\eta(t) - \varphi_\eta(0)]^\alpha = \alpha(t+\eta t^2)$, so dividing by α gives $h_\eta(t) = t + \eta t^2$ and differentiating yields $q_\eta(t) = 1 + 2\eta t$. Since $0 \leq t \leq 1$ and $\eta \geq 0$, we have $1 \leq q_\eta(t) \leq 1 + 2\eta$. This choice ensures $q_\eta \geq 1$, so all hypotheses of Theorems 3.1 and 6.1 apply. It also demonstrates that

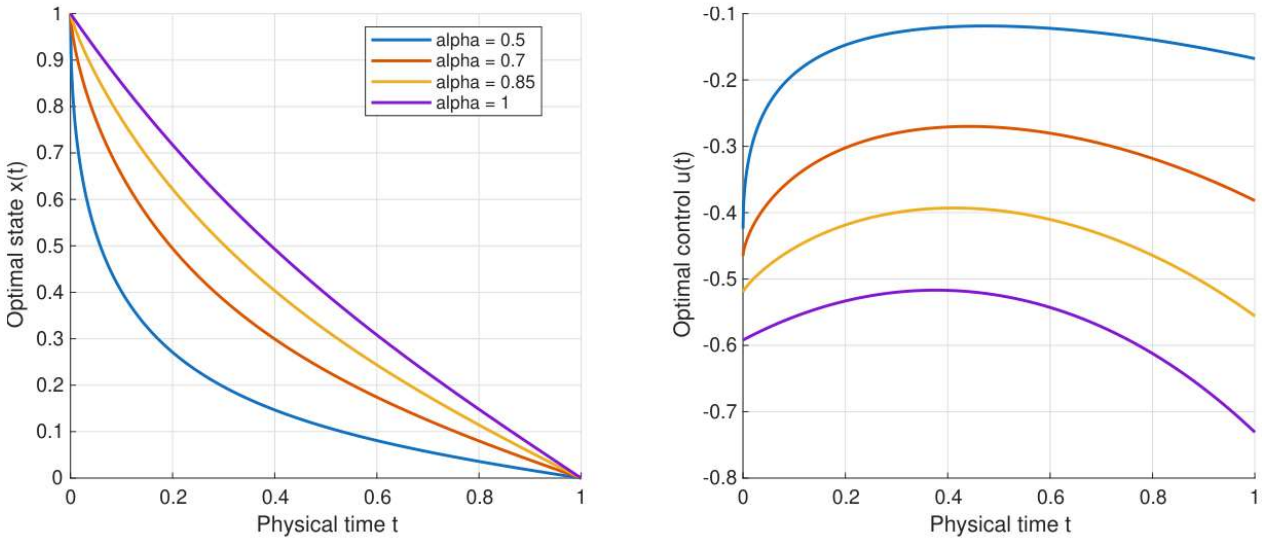

Figure 1: State and control from the exact fixed-endpoint formulas (50)–(51).

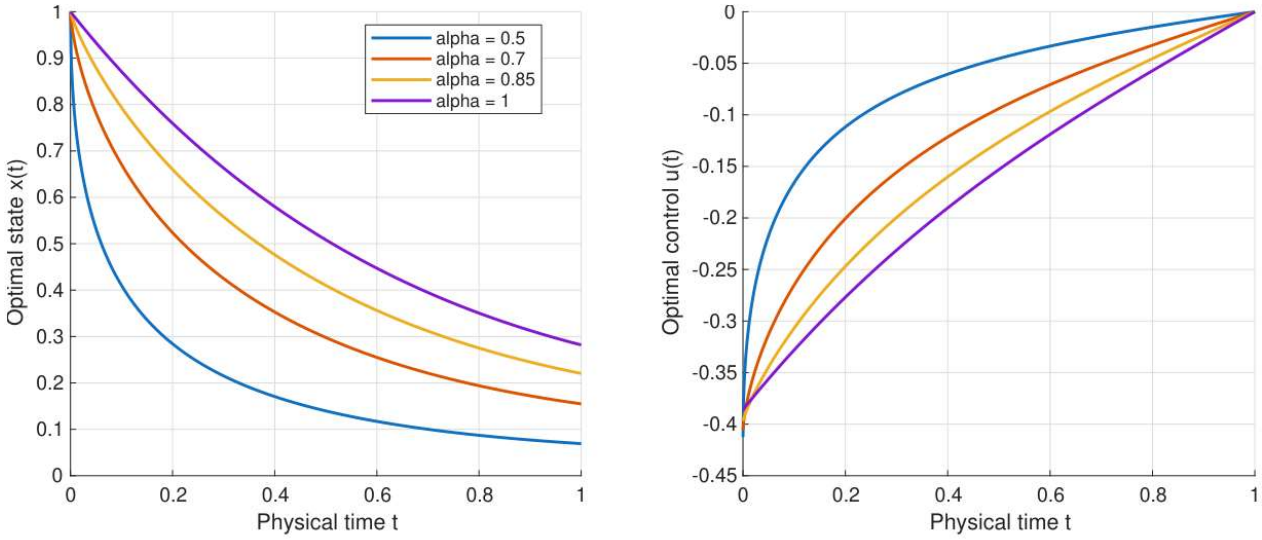

Figure 2: State and control from the exact free-endpoint formulas (53)–(54).

a suitable φ can cancel the nominal α dependence of the local clock: changes in order alone need not alter the dynamics. In clock time $\tau = h_\eta(t)$, $S = 1 + \eta$, and

$$t = \frac{\sqrt{1+4\eta\tau} - 1}{2\eta} \ (\eta > 0), \quad w_\eta(\tau) = \frac{1}{\sqrt{1+4\eta\tau}}, \quad (w_0 \equiv 1). \tag{58}$$

The stationary physical-time canonical system is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} -1 & -1/w_\eta(\tau) \\ -w_\eta(\tau) & 1 \end{pmatrix}\begin{pmatrix} y \\ p \end{pmatrix}, \qquad v(\tau) = -p(\tau)/w_\eta(\tau). \tag{59}$$

For the fixed endpoint $y(S) = 0$ and for the free endpoint $p(S) = 0$, respectively, the unique initial costate can be determined by a 2×2 fundamental matrix, just as in Theorem 5.1, but with a time-dependent propagator.

Proposition 9.1 (Construction and numerical verification of the physical-time propagator). Let $F_\eta(\tau)$ be the unique matrix solution of

$$F_\eta'(\tau) = M_\eta(\tau)F_\eta(\tau), \quad F_\eta(0) = I_2, \quad M_\eta(\tau) = \begin{pmatrix} -1 & -1/w_\eta(\tau) \\ -w_\eta(\tau) & 1 \end{pmatrix}.$$

Write $F_\eta(S) = (f_{ij})_{i,j=1}^2$. For $y(0) = 1$, the initial costate is

$$p_0^{\text{fixed}} = -\frac{f_{11}}{f_{12}}, \qquad p_0^{\text{free}} = -\frac{f_{21}}{f_{22}}. \tag{60}$$

Both denominators are nonzero. With either value, the optimal trajectories are $(y(\tau), p(\tau))^{\mathrm{T}} = F_\eta(\tau)(1, p_0)^{\mathrm{T}}$ and $v(\tau) = -p(\tau)/w_\eta(\tau)$.

Proof. The entries of M_η are continuous on $[0, S]$ because $w_\eta = (1 + 4\eta\tau)^{-1/2} > 0$. Standard successive approximations for a linear system produce a unique fundamental matrix F_η. The terminal state is $y(S) = f_{11} + f_{12}p_0$. Imposing $y(S) = 0$ gives the first formula when $f_{12} \neq 0$. The free terminal costate is $p(S) = f_{21} + f_{22}p_0$, giving the second formula when $f_{22} \neq 0$. To check $f_{12} \neq 0$, assume $f_{12} = 0$. Then the nontrivial canonical solution initialized with $(y(0), p(0)) = (0,1)$ has $y(S) = 0$. Its control satisfies the stationary conditions for a zero-to-zero problem, hence must equal the unique global minimizer $v = 0$. Thus $y = 0$ and $p = -wv = 0$, contradicting $p(0) = 1$. For f_{22}, assume $f_{22} = 0$; the same nonzero initial costate gives $p(S) = 0$. The sufficiency theorem makes its control the unique optimizer for the free-terminal problem with $y(0) = 0$, namely the zero control. Again $p = 0$ contradicts its initial value. Therefore both denominators are nonzero. Finally, the vector formula solves (59) and satisfies the stated endpoints by construction. The stationarity relation determines v, and global optimality and uniqueness follow from Theorem 4.1. In physical time, recover $x(t) = y(h_\eta(t))$ and $u(t) = v(h_\eta(t))$.

The identity

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(yp) = -w_\eta(y^2 + v^2), \qquad J^*_{\text{phys}} = \frac{1}{2}p(0) \tag{61}$$

serves as an independent a posteriori check for either of these particular endpoint cases.

Proof of the physical-time value identity. With $p' = -w_\eta y + p$ and $y' = -y - p/w_\eta$, the ordinary product rule gives

$$\begin{aligned}(yp)' &= y'p + yp' \\ &= (-y - p/w_\eta)p + y(-w_\eta y + p) \\ &= -w_\eta y^2 - p^2/w_\eta = -w_\eta(y^2 + v^2),\end{aligned}$$

where $p = -w_\eta v$. Integrating and noting that $y(0) = 1$ gives $\int_0^S w_\eta(y^2 + v^2)\mathrm{d}\tau = p(0) - y(S)p(S)$. For a fixed terminal state $y(S) = 0$; for a free terminal state $p(S) = 0$. In either case the final product vanishes, so the physical-time cost equals $p(0)/2$. This check does not depend on a particular numerical ODE solver. □

Table 2: Physical-time minimum values with (57). Values obtained by high-accuracy integration of (59) and checked with (61).

η	0	0.25	0.50	1.00
Fixed final state	0.29594583	0.21426842	0.17848360	0.14786196
Free final state	0.19290930	0.17660617	0.16386342	0.14536900

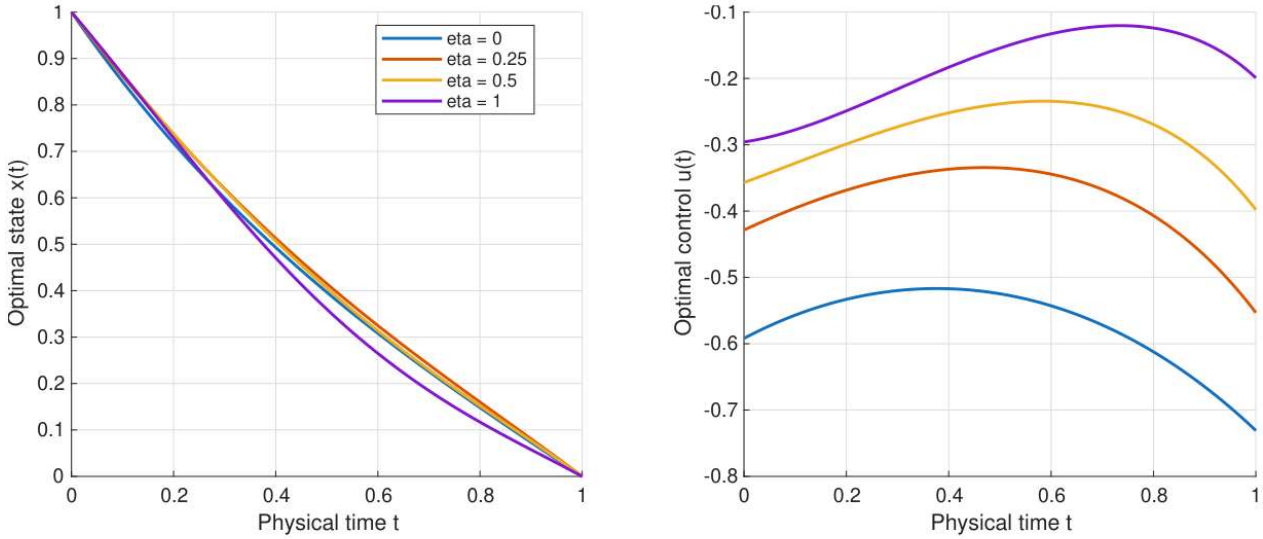

Figure 3: Physical-time optimal trajectories with a fixed terminal state for different η; each curve solves (59).

Numerical implementation

The cost variation in Table 2 cannot be inferred by merely substituting α into a conventional conformable formula. Note that changing η also changes the operational horizon S; these calculations demonstrate model sensitivity, not an isolated causal effect of the fractional order. The analytical formulas and the stated numerical construction reproduce all four figures, both tables, and checks the endpoint conditions and the value identities.

10. Proof dependencies and precise

In order to facilitate reusability, the correct sequence is as follows. Lemma 2.1 provides proof of the integrability of the clock density function. Proposition 2.3 provides proof of the differentiation and inversion formulas from the definition itself. Theorem 2.4 establishes the change of variables formula, while Proposition 2.7 provides the proof of elementary calculus formulae and Proposition 2.8 gives the proof of linear solutions. Theorem 3.1 provides existence and uniqueness theorem for uniformly coercive case, and finally Theorem 4.1 provides the necessary and sufficient optimality conditions. There are three hypotheses which must not be weakened indirectly. Firstly, the differential identity ${}_a\mathcal{C}^{\alpha,\varphi} f = f'/q$ is employed only at those points where the stated ordinary derivative

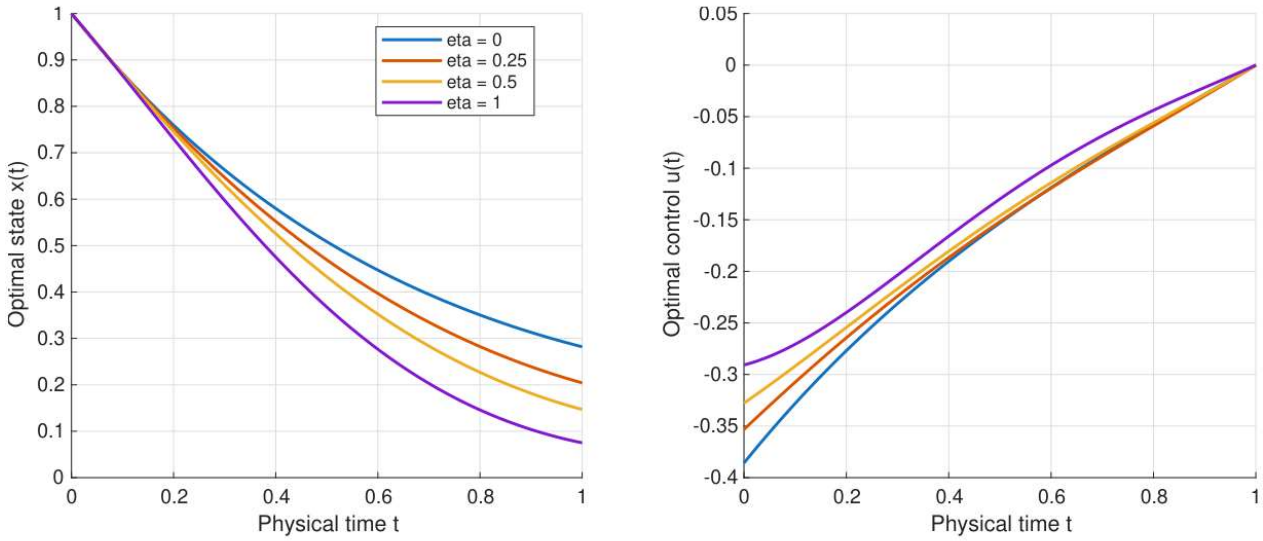

Figure 4: Physical-time optimal trajectories with a free terminal state; the optimal control vanishes at terminal time.

exists; in case of mere measurability of the data, the inverse integral identity is valid almost everywhere, but not everywhere. Secondly, the physical-time LQ existence theorem and Riccati theorem presume boundedness and positivity of w and $1/w$. Thirdly, the Hilbert space Gramian necessitates $q \in L^2(a,T)$; the problem with this condition is remedied by the separate theorem of nonattainment, and not by the extension of the Gramian identity beyond its applicability domain.

11. Discussion and limitations

A few distinctions have to be made to provide a scientifically justified interpretation. First, for regular enough φ, φ-conformable dynamic systems are isomorphic to classical differential equations in clock $h(t)$. This is the reason why matrix exponentials, rather than Mittag-Leffler functions, arise in the autonomous natural clock case. Second, an arbitrary substitution of φ changes the time-parametrization of the state, but does not necessarily introduce new memory effects. Third, the running measure is an intrinsic part of the mathematical model: for $q(t)\mathrm{d}t$ and $\mathrm{d}t$ one gets different stationarity conditions and different (generally speaking) Riccati equations. Fourth, when q vanishes or goes to infinity at a, the physics-time case may require weighted function spaces; the proof of Theorem 3.1 deliberately uses uniform boundedness assumptions rather than relying on Hilbert space theory tricks. Finally, the state constraint $u(t) \in U$, where U is closed and convex set, changes problem (21) into pointwise weighted projection; the corresponding matrixexponential solution does not exist anymore; we do not claim it.

12. Conclusion

The exact operational-time reduction gives a rigorous foundation for φ-conformable optimal control. Under a natural conformable performance measure, fixed- and free-terminal-time-state linear–quadratic problems admit the same canonical matrix-exponential solutions as conventional LQ theory, with physical time replaced by $h(t)$. Under a physical-time measure, the transformed problem is a weighted LQ control problem with a nonautonomous Riccati equation and comparison bounds. Two exact examples, a sharp singular-clock nonattainment theorem, and a physically weighted numerical example verify the formulas and make the

dependence on the modelled clock explicit. A claim of publication-level novelty requires further comparison with the literature on deterministic time transformations, weighted LQ optimal control, and recent numerical conformable optimal-control algorithms [5, 6], beyond the specific results proved here.

Data availability. No external datasets were used. The values in Tables 1–2 and Figures 1–4 were obtained from the formulas and numerical procedures described in this manuscript.

Conflict of interest. To be confirmed by the author before submission.

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