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Original Research Article

Existence and uniqueness for a tempered Ψ-Caputo equation

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Abstract: We study a nonlinear initial value problem for the tempered Ψ-Caputo derivative of order $0 < \alpha < 1$. A continuous right-hand side of at most linear growth yields a solution on every prescribed finite interval. A Lipschitz condition on bounded sets, uniform in time, gives uniqueness on the same interval. The proofs apply Schauder's theorem and Banach's contraction principle to a single Volterra operator. Two explicit examples give, respectively, infinitely many solutions with different waiting times and a unique solution for a nonlinearity with unbounded derivative.

Keywords: tempered Ψ-Caputo derivative; Existence and uniqueness; fixed point theorem.

1. Introduction:

The tempered Ψ-Caputo derivative measures the elapsed interval from s to t by $\Psi(t) - \Psi(s)$. In the associated integral, this increment enters both the power-law kernel and its exponential damping factor. We consider the scalar initial value problem

$$\mathcal{D}^{\alpha}_{\Psi,\lambda} u(t) = f\big(t, u(t)\big), \qquad 0 < t \le T, \qquad u(0) = u_0.$$

Here $0 < \alpha < 1$, $\lambda \ge 0$, and Ψ is continuously differentiable with $\Psi(0) = 0$ and $\Psi' > 0$. The question is which assumptions on f guarantee a solution throughout $[0, T]$, and which additional assumption makes it unique.

Medveď and Brestovanská [1] studied Cauchy problems for the tempered Ψ-Caputo derivative. Mali et al. [2] developed the associated operator calculus and applied Banach's principle to nonlinear equations. Medveď and Pospíšil [3] used a generalized Laplace transform for a linear inhomogeneous problem. For nonlinear initial value problems, Pospíšil and Pospíšilová Škripková [4] established local uniqueness results that also allow derivatives or integrals of the state on the right-hand side. More recent work treats existence for hybrid equations with tempered ψ-Caputo operators [5] and solvability of boundary value problems for systems with the tempered Caputo derivative [6]. In reference [9], the authors investigated implicit FDEs with anti-periodic boundary conditions, while references [10,11] applied fractional operators to the dynamics of epidemics, and socio-economic modeling, respectively.

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For the scalar problem above, we give a proof on the prescribed interval that uses two exponential weights. The first determines a bounded set containing every solution from the growth data alone. After that set is fixed, the second weight makes the integral operator contractive using only the Lipschitz constant on the relevant range. This order of choices avoids a restriction on the interval length. The two examples show why the distinction matters even for continuous nonlinearities of at most linear growth.

2. Fractional integrals and derivatives

Throughout, let $T > 0$, $S = \Psi(T)$, $x = \Psi(t)$, and $\Delta_{ts} = \Psi(t) - \Psi(s)$. Let $X = C([0,S],\mathbb{R})$ with the supremum norm. Since $\Psi' > 0$, the map $\Psi\colon [0,T] \to [0,S]$ has a continuously differentiable inverse. In the variable x, the left Riemann–Liouville integral of order $\beta > 0$ is [7]

$$\big(J^{\beta}h\big)(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\,h(s)\,ds, \qquad \big(J^{\beta}h\big)(0) = 0.$$

For continuous h, the substitution $s = xv$ gives

$$\big(J^{\beta}h\big)(x) = \frac{x^{\beta}}{\Gamma(\beta)}\int_0^1 (1-v)^{\beta-1}\,h(xv)\,dv.$$

The integrand is bounded in absolute value by $\| h \|_{\infty}\,(1-v)^{\beta-1}$. Dominated convergence gives $J^{\beta}h \in X$.

The change of variable and the exponential weight define an invertible map $U\colon C([0,T],\mathbb{R}) \to X$,

$$(Uu)(x) = e^{\lambda x}u\big(\Psi^{-1}(x)\big), \qquad (U^{-1}z)(t) = e^{-\lambda\ (t)}z\big(\Psi(t)\big).$$

Tempered fractional integral. The integral of order β with respect to Ψ is $I^{\beta}_{\Psi,\lambda} = U^{-1}J^{\beta}U$. Substitution of $y = \Psi(s)$ gives [1, 2]

$$\Big(I^{\beta}_{\Psi,\lambda}g\Big)(t) = \frac{1}{\Gamma(\beta)}\int_0^t e^{-\lambda\Delta_{ts}}\,\Delta_{ts}^{\beta-1}g(s)\Psi'(s)\,ds.$$

Regularized Caputo derivative. For $z \in X$ define $D^{\alpha}_{C}z$ whenever $J^{1-\alpha}\big(z - z(0)\big) \in C^1[0,S]$. We use the corresponding regularized tempered derivative [4],

$$D^{\alpha}_{C}z = \frac{d}{dx}J^{1-\alpha}\big(z - z(0)\big), \qquad \mathcal{D}^{\alpha}_{\Psi,\lambda}u = U^{-1}D^{\alpha}_{C}(Uu).$$

Thus a solution is a continuous u with the prescribed initial value, with Uu in this domain, and with the differential equation satisfied for $0 < t \le T$. The derivative at the left endpoint in the domain condition is one-sided.

Formula for differentiable data. For $u \in C^1[0,T]$, write $\delta_{\Psi}u = u'/\Psi'$. Since $z = Uu$ satisfies $z - z(0) = J^1z'$, the composition rule below gives $D^{\alpha}_{C}z = J^{1-\alpha}z'$. Hence

$$\mathcal{D}^{\alpha}_{\Psi,\lambda}u = I^{1-\alpha}_{\Psi,\lambda}(\delta_{\Psi}u + \lambda u).$$

The regularized definition also applies when the solution has a fractional-power increment at the initial time.

2.1 Composition and inverse identities

Lemma 1. For $\beta, \gamma > 0$ and $h \in X$, the operators satisfy $J^{\beta} J^{\gamma} h = J^{\beta+\gamma} h$. For $0 < \beta < 1$, J^{β} is injective on X.

Proof. With $0 \le r < x$ and $s = r + (x - r)v$, Euler's beta integral gives

$$\int_r^x (x-s)^{\beta-1} (s-r)^{\gamma-1} \, ds = (x-r)^{\beta+\gamma-1} \int_0^1 (1-v)^{\beta-1} v^{\gamma-1} \, dv$$

$$= (x-r)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}.$$

Since $|h(r)| \le \| h \|_{\infty}$,

$$\int_0^x |h(r)| \, (x-r)^{\beta+\gamma-1} \, dr \le \frac{\| h \|_{\infty} \, x^{\beta+\gamma}}{\beta+\gamma} < \infty.$$

By Fubini's theorem,

$$\big(J^{\beta} J^{\gamma} h\big)(x) = \frac{1}{\Gamma(\beta)\Gamma(\gamma)} \int_0^x h(r) \int_r^x (x-s)^{\beta-1} (s-r)^{\gamma-1} \, ds \, dr$$

$$= \frac{1}{\Gamma(\beta+\gamma)} \int_0^x (x-r)^{\beta+\gamma-1} \, h(r) \, dr = \big(J^{\beta+\gamma} h\big)(x).$$

For $0 < \beta < 1$,

$$J^{\beta} h = 0 \;\Rightarrow\; J^{1-\beta} J^{\beta} h = J^{1} h = 0 \;\Rightarrow\; h = (J^{1} h)' = 0. \qquad \blacksquare$$

Lemma 2. For $g \in C[0, T]$ and u in the domain of $\mathcal{D}^{\alpha}_{\Psi,\lambda}$,

$$\mathcal{D}^{\alpha}_{\Psi,\lambda} I^{\alpha}_{\Psi,\lambda} g = g, \qquad I^{\alpha}_{\Psi,\lambda} \mathcal{D}^{\alpha}_{\Psi,\lambda} u = u - e^{-\lambda} \; u(0).$$

Proof. Set $h = Ug$, $z = Uu$, and $w = J^{1-\alpha}\big(z - z(0)\big)$. Since $(J^{\alpha} h)(0) = 0$,

$$D^{\alpha}_{C} J^{\alpha} h = (J^{1-\alpha} J^{\alpha} h)' = (J^{1} h)' = h.$$

Since $w(0) = 0$ and $w' = D^{\alpha}_{C} z$,

$$J^{1-\alpha} J^{\alpha} D^{\alpha}_{C} z = J^{1} w' = w = J^{1-\alpha}\big(z - z(0)\big).$$

By Lemma 1 and conjugation with U,

$$J^{\alpha} D^{\alpha}_{C} z = z - z(0), \qquad U^{-1}\big(z - z(0)\big) = u - e^{-\lambda \Psi} u(0). \qquad \blacksquare$$

Power functions. For $0 \le d < S$ and $\rho > 0$, write $p_{d,\rho}(x) = (x-d)^{\rho}_{+}$, where $r_{+} = \max\{r, 0\}$. The integral vanishes for $x \le d$. For $x > d$, substituting $s = d + (x-d)v$ gives

$$\big(J^{\beta} p_{d,\rho}\big)(x) = \frac{(x-d)^{\rho+\beta}}{\Gamma(\beta)} \int_0^1 (1-v)^{\beta-1} \, v^{\rho} \, dv.$$

Thus $J^{\beta} p_{d,\rho} = \Gamma(1+\rho) p_{d,\rho+\beta} / \Gamma(1+\rho+\beta)$, the identity used to find the waiting-time solutions in Example 1.

3. The equivalent Volterra equation

Let $f: [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Write $z = Uu$ and define

$$F(x,v) = e^{\lambda x} f\big(\Psi^{-1}(x), e^{-\lambda x} v\big), \qquad F_z(x) = F\big(x, z(x)\big).$$

For $z \in X$, $F_z \in X$. The integral operator P associated with the initial condition is

$$(Pz)(x) = u_0 + (J^{\alpha} F_z)(x) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} F\big(s, z(s)\big)\, ds.$$

Proposition 1. The initial value problem has a solution u if and only if $z = Uu$ is a fixed point of P. In the original variable that solution satisfies

$$u(t) = u_0 e^{-\lambda \Psi(t)} + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda \Delta_{ts}}\, \Delta_{ts}^{\alpha-1} f\big(s, u(s)\big) \Psi'(s)\, ds.$$

Proof. If u is a solution, then $z(0) = u_0$ and $D_C^{\alpha} z = F_z$. Hence, by Lemma 2,

$$z - z(0) = J^{\alpha} D_C^{\alpha} z = J^{\alpha} F_z \quad \Rightarrow \quad z = Pz.$$

Conversely, if $z = Pz$, then $z(0) = u_0$ and

$$J^{1-\alpha}\big(z - z(0)\big) = J^{1-\alpha} J^{\alpha} F_z = J^1 F_z \in C^1[0,S].$$

Consequently,

$$D_C^{\alpha} z = F_z, \qquad \mathcal{D}_{\Psi,\lambda}^{\alpha}(U^{-1} z) = U^{-1} F_z = f(\cdot, U^{-1} z).$$

Applying U^{-1} to $z = u_0 + J^{\alpha} F_z$ gives the stated integral equation. ▫

Behavior at the initial point. For any fixed point, let $\omega(x) = \max_{0 \le s \le x} |F_z(s) - F_z(0)|$. Continuity gives $\omega(x) \to 0$ as $x \downarrow 0$. Since $J^{\alpha} 1 = x^{\alpha}/\Gamma(1+\alpha)$,

$$\left| z(x) - u_0 - \frac{F_z(0) x^{\alpha}}{\Gamma(1+\alpha)} \right| \le \frac{\omega(x) x^{\alpha}}{\Gamma(1+\alpha)}.$$

Since $F_z(0) = f(0, u_0)$ and $x = \Psi(t)$,

$$\frac{e^{\lambda \Psi(t)} u(t) - u_0}{\Psi(t)^{\alpha}} \to \frac{f(0,u_0)}{\Gamma(1+\alpha)} \qquad (t \downarrow 0).$$

If $f(0,u_0) \neq 0$, this gives a leading increment proportional to x^{α} in z. Requiring an ordinary derivative at zero would exclude these solutions.

4. Existence by Schauder's fixed point theorem

Theorem 1. Suppose f is continuous and there are constants $a, b \ge 0$ such that

$$|f(t,r)| \le a + b|r|, \qquad 0 \le t \le T, \quad r \in \mathbb{R}.$$

For every $u_0 \in \mathbb{R}$, the initial value problem has at least one solution on $[0,T]$.

Schauder's theorem states that a continuous self-map of a nonempty, closed, convex subset of a Banach space has a fixed point if its image is relatively compact [8].

Proof. Put $A = a e^{\lambda S}$. For $0 \le x \le S$,

$$|F(x,v)| \le e^{\lambda x}\big(a + b e^{-\lambda}\ |v|\big) \le A + b|v|.$$

For $k > 0$, define the equivalent norm

$$\| h \|_k = \max_{0 \le x \le S} e^{-kx} |h(x)|, \qquad e^{-k} \| h \|_\infty \le \| h \|_k \le \| h \|_\infty.$$

Since $|h(s)| \le e^{ks} \| h \|_k$, substituting $r = x - s$ gives

$$e^{-k} |J^\alpha h(x)| \le \frac{\| h \|_k}{\Gamma(\alpha)} \int_0^x r^{\alpha-1} e^{-kr} \, dr.$$

Consequently,

$$\int_0^\infty r^{\alpha-1} e^{-k} \, dr = k^{-\alpha} \Gamma(\alpha), \qquad \| J^\alpha h \|_k \le k^{-\alpha} \| h \|_k.$$

Choose $k > 0$ with $q_0 = bk^{-\alpha} < 1$. Set $C_0 = |u_0| + Ak^{-\alpha}$ and define

$$R = 1 + \frac{C_0}{1 - q_0}, \qquad B = \{z \in X : \| z \|_k \le R\}.$$

Then $0 \in B$, and B is closed and convex. Since $\| F_z \|_k \le A + b \| z \|_k$,

$$\| Pz \|_k \le |u_0| + k^{-\alpha} \| F_z \|_k \le C_0 + q_0 \| z \|_k.$$

Therefore,

$$z \in B \Rightarrow \| Pz \|_k \le C_0 + q_0 R = R - (1 - q_0) < R.$$

Moreover, for every fixed point $w \in X$,

$$Pw = w \Rightarrow (1 - q_0) \| w \|_k \le C_0 \Rightarrow \| w \|_k \le R - 1.$$

4.1 Continuity and compactness on the invariant ball

Put $H = Re^{kS}$, $M = A + bH$, and $Q = [0, S] \times [-H, H]$. For $z \in B$,

$$\| z \|_\infty \le H, \qquad \| F_z \|_\infty \le M, \qquad \| Pz \|_\infty \le |u_0| + \frac{MS^\alpha}{\Gamma(1 + \alpha)}.$$

Since F is uniformly continuous on Q, if $z_n, z \in B$ and $\| z_n - z \|_\infty \to 0$, then

$$\| F_{z_n} - F_z \|_\infty = \max_{0 \le x \le S} \left| F\big(x, z_n(x)\big) - F\big(x, z(x)\big) \right| \to 0.$$

Hence

$$\| Pz_n - Pz \|_\infty \le \frac{S^\alpha}{\Gamma(1 + \alpha)} \| F_{z_n} - F_z \|_\infty \to 0.$$

For $0 \le x < y \le S$, set $K(r) = r^{\alpha-1}/\Gamma(\alpha)$ for $r > 0$. Since K decreases,

$$(Pz)(y) - (Pz)(x) = \int_0^x [K(y - s) - K(x - s)] \, F_z(s) \, ds + \int_x^y K(y - s) F_z(s) \, ds.$$

Therefore,

$$|(Pz)(y) - (Pz)(x)| \le M \int_0^x [K(x - s) - K(y - s)] \, ds + M \int_x^y K(y - s) \, ds.$$

Since $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$,

$$\int_0^x [K(x - s) - K(y - s)] \, ds = \frac{x^\alpha - y^\alpha + (y - x)^\alpha}{\Gamma(1 + \alpha)},$$

$$\int_x^y K(y-s)\,ds = \frac{(y-x)^\alpha}{\Gamma(1+\alpha)}.$$

Consequently,

$$\sup_{z\in B}|(Pz)(y)-(Pz)(x)| \le \frac{2M}{\Gamma(1+\alpha)}(y-x)^\alpha.$$

By Arzelà–Ascoli, $P(B)$ is relatively compact in X. Since $P: B \to B$ is continuous and B is nonempty, closed and convex, Schauder's theorem gives

$$z_* \in B, \qquad Pz_* = z_*.$$

By Proposition 1, $u(t) = e^{-\lambda\Psi(t)}z_*\big(\Psi(t)\big)$ is a solution on $[0,T]$. ▫

5. Uniqueness by Banach's fixed point theorem

Theorem 2. Assume the hypotheses of Theorem 1. Suppose that for every $H > 0$ there is a constant $L_H \ge 0$ such that, for all $t \in [0,T]$ and $|r|, |s| \le H$,

$$|f(t,r) - f(t,s)| \le L_H|r-s|.$$

Then the initial value problem has exactly one solution on $[0,T]$.

Banach's theorem states that a self-map of a nonempty complete metric space with Lipschitz constant strictly less than one has a unique fixed point [8].

Proof. Fix the ball B from Theorem 1 and $H = Re^{kS}$. Choose $\eta > 0$ so that $q = L_H\eta^{-\alpha} < 1$. For $h \in X$,

$$e^{-\eta S}\,\|h\|_\infty \le \|h\|_\eta \le \|h\|_\infty, \qquad \|h\|_k \le e^{\eta S}\,\|h\|_\eta.$$

If $(v_n) \subset B$ is Cauchy in $\|\cdot\|_\eta$, then

$$\|v_n - v_m\|_\infty \le e^{\eta S}\,\|v_n - v_m\|_\eta \to 0.$$

By completeness of X in the supremum norm, $v_n \to v \in X$ uniformly. Hence

$$\|v\|_k = \lim_{n\to\infty}\|v_n\|_k \le R, \qquad \|v_n - v\|_\eta \le \|v_n - v\|_\infty \to 0.$$

Thus (B, d_η) is complete for $d_\eta(z,w) = \|z-w\|_\eta$.

For $z, w \in B$, both $e^{-\lambda x}z(x)$ and $e^{-\lambda x}w(x)$ lie in $[-H,H]$. Therefore,

$$|F_z(x) - F_w(x)| \le e^{\lambda x}L_H e^{-\lambda}\,|z(x)-w(x)| = L_H|z(x)-w(x)|.$$

By the fractional integral estimate with $k = \eta$,

$$d_\eta(Pz, Pw) \le \eta^{-\alpha}\,\|F_z - F_w\|_\eta \le q\,d_\eta(z,w).$$

Since $P(B) \subset B$, Banach's theorem gives a unique fixed point $z_* \in B$. Explicitly, for $z_0 \in B$, $z_{n+1} = Pz_n$, and integers $m > n \ge 0$,

$$d_\eta(z_{n+1}, z_n) \le q^n d_\eta(z_1, z_0),$$

$$d_\eta(z_m, z_n) \le \sum_{j=n}^{m-1} q^j\, d_\eta(z_1, z_0) \le \frac{q^n}{1-q}\,d_\eta(z_1, z_0).$$

Hence $z_n \to z_* \in B$ and $Pz_* = \lim Pz_n = z_*$. For any fixed point $w \in X$, the estimate in Theorem 1 gives $\| w \|_k \leq R - 1$, so $w \in B$. Consequently,

$$d_\eta(w, z_*) = d_\eta(Pw, Pz_*) \leq q\, d_\eta(w, z_*) \quad \Rightarrow \quad w = z_*.$$

Proposition 1 gives uniqueness on the entire interval $[0, T]$. ▫

6. Examples

Example 1 with a waiting time

Let $T = 1$, $\Psi(t) = \log(1 + t)$, $\alpha = 1/2$, $\lambda = 1$, and $u_0 = 0$. Then $\Psi'(t) = 1/(1 + t) > 0$, $S = \log 2$, and the right-hand side

$$f(t, r) = \sqrt{\frac{|r|}{1 + t}}$$

is continuous. Since $2\sqrt{|r|} \leq 1 + |r|$, Theorem 1 applies with $a = b = 1/2$. The local Lipschitz condition at $r = 0$ fails, since

$$\frac{|f(0, r) - f(0,0)|}{|r|} = \frac{1}{\sqrt{|r|}} \to \infty \qquad (r \to 0).$$

Here $F(x, v) = \sqrt{|v|}$. For $z_d(x) = c(x - d)_+$ with $c > 0$ and $0 \leq d < \log 2$, the power identity gives

$$z_d = J^{1/2}\sqrt{z_d} \quad \Leftrightarrow \quad c = \sqrt{c}\,\frac{\Gamma(3/2)}{\Gamma(2)} \quad \Leftrightarrow \quad c = \frac{\pi}{4}.$$

With this value, take $d \in [0, \log 2]$. Both z_d and $J^{1/2}\sqrt{z_d}$ vanish for $x \leq d$. For $x > d$,

$$\left(J^{1/2}\sqrt{z_d}\right)(x) = \frac{1}{2}\int_d^x \frac{\sqrt{s - d}}{\sqrt{x - s}}\, ds = \frac{x - d}{2}\int_0^1 \frac{\sqrt{v}}{\sqrt{1 - v}}\, dv = \frac{\pi}{4}(x - d).$$

Consequently, $z_d = Pz_d$. The corresponding solutions are

$$u_d(t) = \frac{\pi}{4(1 + t)}(\log(1 + t) - d)_+.$$

For $d < \log 2$, the solution first becomes positive immediately after $t = e^d - 1$. Distinct values of d give distinct solutions, and $d = \log 2$ gives the zero solution.

Example 2 with a locally Lipschitz nonlinearity

Let $T = 1$, $\Psi(t) = t + t^2$, $\alpha = 2/3$, $\lambda = 1/2$, and $u_0 = \pi$. Then $\Psi'(t) = 1 + 2t > 0$ and $x = \Psi(t) \in [0,2]$. Take $f(t, r) = r\sin\left(e^{x/2}r\right)$. For $|r| \leq H$,

$$|f(t, r)| \leq |r|, \qquad |\partial_r f(t, r)| = \left|\sin\left(e^{x/2}r\right) + e^{x/2}r\cos\left(e^{x/2}r\right)\right| \leq 1 + eH.$$

The mean value theorem gives $L_H = 1 + eH$, and the growth bound holds with $a = 0$, $b = 1$. Since $\partial_r f(0, 2\pi n) = 2\pi n$, a global Lipschitz constant does not exist. Here $F(x, v) = v\sin v$, so $F(x, \pi) = 0$ and $z(x) = \pi$ satisfies $z = Pz$. Theorem 2 gives the unique solution

$$u(t) = \pi\exp\left(-\frac{t + t^2}{2}\right), \qquad 0 \leq t \leq 1.$$

7. Conclusion

For the regularized tempered Ψ-Caputo initial value problem, continuity and linear growth suffice for existence on any prescribed finite interval. The growth bound first gives an invariant ball containing every solution. The weakly singular integral maps this ball into an equicontinuous family, so Schauder's theorem applies. If the nonlinearity is uniformly Lipschitz on bounded state intervals, changing the norm on the same ball yields a Banach contraction. The waiting-time family and the explicit exponential solution show how these two sets of assumptions affect a given initial value.

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